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JUSTIFIED SEQUENCES IN STRING DIAGRAMS: A COMPARISON BETWEEN TWO APPROACHES TO CONCURRENT GAME SEMANTICS

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Abstract. We compare two approaches to concurrent game semantics, one by Tsukada and Ong for a simply-typed λ-calculus and the other by the authors and collaborators for CCS and the π-calculus. The two approaches are obviously related, as they both define strategies as sheaves for the Grothendieck topology induced by embedding "views" into "plays". However, despite this superficial similarity, the notions of views and plays differ significantly: the former is based on standard justified sequences, the latter uses string diagrams.

In this paper, we relate both approaches at the level of plays. Specifically, we design a notion of play (resp. view) for the simply-typed λ-calculus, based on string diagrams as in our previous work, into which we fully embed Tsukada and Ong's plays (resp. views). We further provide a categorical explanation of why both notions yield essentially the same model, thus demonstrating that the difference is a matter of presentation.

In passing, we introduce an abstract framework for producing sheaf models based on string diagrams, which unifies our present and previous models.

Contents

1. Introduction

Two approaches to concurrent game semantics. Recent advances in concurrent game semantics have produced new games models for CCS and the π-calculus [18, 19, 7] and a non-deterministic, simply-typed λ-calculus [31]. These models are based on categories of innocent and concurrent strategies which share the feature of being defined as categories of *sheaves* over a site of plays.

The two models are obviously related, as they both define innocent strategies as sheaves for the Grothendieck topology induced by embedding views into plays. However, despite this superficial similarity, the notions of views and plays differ significantly. Indeed, Tsukada and Ong define them as *justified sequences* of moves satisfying additional conditions, as in standard Hyland-Ong/Nickau (HON) game-semantics [20, 27]. On the other hand, our plays [18, 19, 7] are based on *ad hoc* string diagrams, as originally suggested by Melliès in a different setting (*circa* 2008, published as Melliès [26]).

This appears to have motivated one of the main – legitimate – objections to our approach: plays are not justified sequences, so why call this game semantics?

Comparing both approaches. In this paper, we relate both approaches not only at the superficial level described above, but also at the level of plays. Specifically, we design a notion of play (resp. view) for the simply-typed λ-calculus, based on string diagrams as in our previous work, into which we embed Tsukada and Ong's plays (resp. views) as justified sequences. We thus obtain (for each pair of *arenas* [20]) a commuting square

$$\begin{tikzcd} \mathbb{V}_{A,B} \arrow[r, hook, "\mathsf{i}_{TO}"] \arrow[d, "F^{\mathbb{V}}"'] & \mathbb{P}_{A,B} \arrow[d, "F"] \\ \mathbb{E}^{\mathbb{V}}(A \vdash B) \arrow[r, hook, "\mathsf{i}"'] & \mathbb{E}(A \vdash B) \end{tikzcd} \quad (1)$$

of embeddings of categories:

- i_{TO} denotes the embedding of Tsukada and Ong's views (TO-views for short) into plays (TO-plays for short);
- i denotes the embedding of our views into plays;
- $F^{\mathbb{V}}$ and F respectively denote the constructed embeddings from TO-views into views and from TO-plays into plays.

Furthermore, we prove that all these embeddings are full (Theorem 144), and that $F^{\mathbb{V}}$ is an equivalence of categories (Theorem 151).

Using this and Guitart's theory of *exact squares* [16], we provide a categorical explanation of why both induced categories of innocent strategies coincide.

Terminology 1. *Let us make the following point precise now, for clarity. In standard game semantics as well as in our approach, there are in fact two notions of innocent strategies:*

- *The first is as sets of* views *with additional properties. In a concurrent setting, this generalises to presheaves on views. We here call these* behaviours *(resp.* TO-behaviours *for Tsukada and Ong's notion).*

- *The second is as sets of* plays *with additional properties, including so-called innocence. In a concurrent setting, this generalises to sheaves on views. We here call these* innocent strategies *(resp.* innocent TO-strategies*). Mere presheaves on plays thus correspond to possibly non-innocent strategies.*

Our second result (Corollary 116) states that the square of functors

$$\begin{tikzcd} \widehat{\mathbb{V}_{A,B}} \arrow[r, hook, "\Pi_{\mathrm{i}_{TO}}"] & \widehat{\mathbb{P}_{A,B}} \\ \mathbb{E}^{\mathbb{V}}\widehat{(A \vdash B)} \arrow[u, "\Delta_{F^{\mathbb{V}}}"] \arrow[r, hook, "\Pi_{\mathrm{i}}"'] & \mathbb{E}\widehat{(A \vdash B)} \arrow[u, "\Delta_F"'] \end{tikzcd}$$

induced by (1) commutes up to isomorphism. Here, $\prod_f$ denotes right Kan extension along f^{op}, and Δ_f denotes restriction along f^{op}. The result entails:

- that $\Delta_{F^{\mathbb{V}}}$ is an equivalence of categories between TO-behaviours and behaviours;
- that Δ_F restricts to an equivalence of categories between innocent TO-strategies and innocent strategies.

Remark 1. *We are perhaps more honest than needed in claiming only a full embedding from TO-plays into plays. Indeed, it would be very easy to impose an additional condition on plays – alternation – in order to obtain an equivalence of categories. The point is that we want to compare the purely diagrammatic notion of play with the classical one. And since we obtain an equivalence between both notions of innocent strategies anyway, we feel the result is in fact more convincing.*

In passing, we provide an alternative characterisation of our, and hence also of Tsukada and Ong's, categories of views and plays for HON games as subcategories of slices of a presheaf category (Theorem 98). This yields a presentation of views and plays which, though less elementary, offers perhaps more structure than the original and thus may be of interest for future developments.

In summary, our main contribution is a clarification of the link between our plays based on string diagrams and the more classical justified sequences: the difference is essentially a matter of presentation.

An abstract framework for constructing sheaf models. As a secondary contribution, when constructing our model, instead of merely applying previously used methods in a slightly different context, we set up an abstract framework for producing sheaf models based on string diagrams. This framework admits both of our previous models and the new one presented here as special cases. The hope is of course that this will facilitate the construction of future sheaf models. Let us briefly explain this before delving into technical details.

In both our approach and Tsukada and Ong's, there is in fact one category of plays $\mathbb{E}_X$ for each initial position X in the game (as we saw above with $X = (A \vdash B)$). In our approach, these categories are constructed in a uniform way from an algebraic structure called a *pseudo double category* [8, 9, 13, 14, 24, 12], which, intuitively, describes the game as a whole. A pseudo double category essentially consists of a set ob($\mathbb{D}$) of *objects*, modelling positions in the game. For each pair (X, Y) of positions, there is a set $\mathbb{D}_v(Y, X)$ which models all *plays* with initial position X and final position Y. Moreover, for all positions X' there is another set $\mathbb{D}_h(X', X)$ of *morphisms* from X' to X, which, roughly, models all ways of embedding the position X' into X. Finally, given plays and morphisms as in

$$\begin{tikzcd} Y' \arrow[r, "h"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "k"'] & X \end{tikzcd}$$

(where plays in $\mathbb{D}_v$ are marked with bullets to distinguish them from morphisms in $\mathbb{D}_h$), there is a set $\mathbb{D}(h, u, k, v)$ of *cells*, which model embeddings from u into v, preserving both the initial and final positions. E.g., u could describe the part of the play v which concerns players in X'. Pseudo double categories enjoy quite a few bits of additional structure, like composition of morphisms and plays, and vertical and horizontal composition of cells, satisfying coherence conditions. The pseudo double categories used in our sheaf models satisfy a few additional properties, among which the crucial "fibredness" of the title.

Beyond providing a uniform construction for the categories $\mathbb{E}_X$, this rich structure yields links between them, which greatly facilitate the development. Indeed, each play $u: Y \twoheadrightarrow X$ induces a functor $\mathsf{S}(u): \mathrm{Sh}(\mathbb{E}_X) \to \mathrm{Sh}(\mathbb{E}_Y)$ between the corresponding categories of sheaves (a.k.a. innocent strategies). Similarly, each morphism $h: X' \to X$ induces a restriction functor $\mathsf{S}(h): \mathrm{Sh}(\mathbb{E}_X) \to \mathrm{Sh}(\mathbb{E}_{X'})$. These functors are semantically relevant: for any innocent strategy $S \in \mathrm{Sh}(\mathbb{E}_X)$, $\mathsf{S}(u)(S)$ denotes the "residual" of S after u, i.e., a description of how the players of X would behave according to S after playing u. A meaningful transition system on innocent strategies is then (roughly) given by triples $S \xrightarrow{M} \mathsf{S}(M)(S)$, for any move M in the game. The obtained transition system is shown in both of our previous models to be closely related to the operational semantics of the considered calculus. On the other hand, $\mathsf{S}(h)(S)$ denotes a *restriction* of S to X', i.e., a description of S for players in the subposition X' of X. This is useful for composing strategies: if a given position X is divided into two subpositions X' and X'', thought of as two teams, a composite (in a sense analogous to parallel composition in game semantics) of $S' \in \mathrm{Sh}(\mathbb{E}_{X'})$ and $S'' \in \mathrm{Sh}(\mathbb{E}_{X''})$ is any $S \in \mathrm{Sh}(\mathbb{E}_X)$ whose respective restrictions to X' and X'' give S' and S''.

We thus set out to produce such fibred pseudo double categories automatically from more basic data. The kind of basic data we will use is the notion of *signature* defined in Section 3. From any *signature* S, we construct a pseudo double category $\mathbb{D}(\mathsf{S})$. Up to the verification of a few additional conditions on $\mathbb{D}(\mathsf{S})$, new sheaf models may thus be produced directly from nice signatures S.

As mentioned above, one crucial such condition, on which the very construction of our categories $\mathbb{E}_X$ is based, is fibredness. Our second main contribution is then in Section 4 to prove that under suitable hypotheses, $\mathbb{D}(\mathsf{S})$ satisfies this property. More precisely, we provide two results:

- Under a necessary and sufficient, but hard-to-verify condition essentially saying that fibredness is satisfied for all basic moves in S, we prove that $\mathbb{D}(\mathsf{S})$ is fibred (Theorem 83).
- We then exhibit sufficient, easier-to-verify conditions for the result to hold (Theorem 92).

By plugging both results together, we obtain (Corollary 93) that any S satisfying the given sufficient conditions yields a fibred $\mathbb{D}(\mathsf{S})$.

Plan. The paper is organised as follows. We start by recalling Tsukada and Ong's notion of strategy, the notion of pseudo double category, as well as a few useful facts about presheaf categories in Section 2. In Section 3, we first design what will be the signature S_{HON} for HON games. We then generalise this to define signatures and construct the pseudo double category $\mathbb{D}(\mathsf{S})$ of plays on any signature S, which we finally instantiate with S_{HON}, giving rise to the pseudo double category $\mathbb{D}(\mathsf{S}_{HON})$

we want to construct for HON games. We also state and motivate the fibredness property, which we then study in Section 4, where we provide sufficient criteria for it to be satisfied by $\mathbb{D}(\mathsf{S})$. In particular, we show that these criteria are satisfied by S_{HON}. In Section 5, we relate the obtained categories of views, plays and strategies to Tsukada and Ong's. We finally conclude in Section 6.

Notation. We often use natural numbers n to denote sets $\{1, \ldots, n\}$. We denote by $\widehat{\mathcal{C}}$ the category of presheaves over any category $\mathcal{C}$, i.e., contravariant functors to the category Set of sets. We further denote $F(f)(x)$ by $x \cdot f$, for any presheaf $F \in \widehat{\mathcal{C}}$, $C, D \in \mathcal{C}$, $f : C \to D$, and element $x \in F(D)$. The Yoneda embedding is denoted by $\mathsf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ and often left implicit. Finally, for any category $\mathcal{C}$, let $\mathcal{C}_f$ denote its full subcategory of finitely presentable objects [1].

2. Preliminaries

2.1. **Tsukada-Ong strategies.** Let us start with a brief recapitulation on Tsukada and Ong's categories of views and plays, as well as their notion of strategy.

As usual in game semantics, games are based on *arenas*. An arena consists of a finite set of polarised moves, organised into a forest via a so-called *justification* relation. A compact definition is:

Definition 2. *An* arena *is a simple forest, i.e., a directed, simple graph in which all vertices are uniquely reachable from a unique root (= vertex without a parent).*

Vertices will be called *moves*, and roots will be deemed *initial*.

Remark 2. *This graph-based notion of arena, already used, e.g., by Harmer et al. [17], slightly differs from Tsukada and Ong's relation-based notion. For instance, their notion comprises non-empty arenas without initial moves, e.g., the integers with justification given by predecessor and so-called ownership given by parity. In fact, any relation-based arena A is isomorphic (in the category of arenas and innocent strategies) to the simple forest F_A given by the part of A which is reachable from its roots. Furthermore, F_A and A yield the exact same categories of views and plays. So we deliberately restrict attention to graph-based arenas.*

Notation 3. *All forests considered in the sequel are simple, and we omit to mention it. We denote by $\sqrt{A}$ the set of roots of A. If A is an arena and m is a move in A, then $A_{/m}$ is the forest strictly below m, and $A \cdot m$ denotes $A_{/m}$ when $m \in \sqrt{A}$. Any forest A is a coproduct of trees, so that $A \cong \sum_{m \in \sqrt{A}} T_m$ where each T_m is a tree. For any arena A and $m \notin A$, we denote by $m.A$ the unique tree T such that $\sqrt{T} = \{m\}$ and $T \cdot m = A$. Thus, any forest may be written as $A = \sum_{m \in \sqrt{A}} m.(A \cdot m)$. The* ownership *of any vertex $m \in A$ is O (for* Opponent*) if the length of the unique path from a root to m is even, and P (for* Proponent*) otherwise. So, e.g., all roots have ownership O. We denote this map $M_A \to \{P, O\}$ by λ_A, where M_A is the set of moves of A.*

Let us fix arenas A and B. Let $A \multimap B$ denote the simple graph obtained by adding to $A + B$ an edge $b \to a$ for all $b \in \sqrt{B}$ and $a \in \sqrt{A}$. The notion of ownership straightforwardly extends to $A \multimap B$ since all paths from any root to some vertex v have the same length. Concretely, ownership is left unchanged in B but reversed in A (if B contains at least one root).

Remark 3. *This is an arena iff B has at most one root or A is empty.*

Definition 4. *A* justified sequence *on (A, B) consists of a natural number $n \in \mathbb{N}$, equipped with maps $f : n \to M_A + M_B$ and $\varphi : n \to \{0\} \uplus n$ such that, for all $i \in n$,*

- *$\varphi(i) < i$,*

- *if* $\varphi(i) = 0$ *then* $f(i) \in \sqrt{B}$*, and*
- *if* $\varphi(i) \neq 0$*, then* $f(\varphi(i))$ *is the parent of* $f(i)$ *in* $A \multimap B$*.*

We denote by $|(n, f, \varphi)|$ the length n of the sequence (n, f, φ).

For any $i \in n$, the *view* $\lceil (n, f, \varphi) \rceil_i$ *of* i *in* (n, f, φ) is the subset of n defined inductively by:

- $\lceil (n, f, \varphi) \rceil_i = \{i\}$ if i is an Opponent move with $\varphi(i) = 0$,
- $\lceil (n, f, \varphi) \rceil_i = \lceil (n, f, \varphi) \rceil_j \cup \{i\}$ if i is an Opponent move with $\varphi(i) = j > 0$,
- $\lceil (n, f, \varphi) \rceil_i = \lceil (n, f, \varphi) \rceil_{i-1} \cup \{i\}$ if i is a Proponent move.

A justified sequence $s = (n, f, \varphi)$ on (A, B) is *P-visible* iff for all Proponent moves i, $\varphi(i) \in \lceil s \rceil_i$. We further say that s is *alternating* iff for all $i \in n - 1$, $\lambda_{A \multimap B}(i) \neq \lambda_{A \multimap B}(i + 1)$.

Definition 5. *A* preplay *on the pair of arenas* (A, B) *is a P-visible, alternating, justified sequence on* (A, B)*.*

A morphism of preplays $g: (n, f, \varphi) \to (n', f', \varphi')$ is an injective map $g: n \to n'$ such that:

- $f'(g(i)) = f(i)$ for all $i \in n$,
- $\varphi'(g(i)) = g(\varphi(i))$ for all $i \in n$ (with the convention that $g(0) = 0$),
- $g(2i) = g(2i - 1) + 1$ for all $i \in n/2$.

The last condition says that g should preserve blocks of an Opponent move and the next Proponent move (so-called *OP-blocks*).

Proposition 6. *Pre-plays and morphisms between them form a category* $\mathbb{PP}_{A,B}$*, with composition given by composition of underlying maps.*

Definition 7. *A* TO-play *is a preplay of even length. The category* $\mathbb{P}_{A,B}$ *is the full subcategory of* $\mathbb{PP}_{A,B}$ *spanning TO-plays.*

Definition 8. *A* TO-view *on* (A, B) *is a TO-play* $s = (n, f, \varphi)$ *such that* $|s| > 0$ *and* $\lceil s \rceil_n = s$*. Let* $\mathbb{V}_{A,B}$ *denote the full subcategory of* $\mathbb{P}_{A,B}$ *spanning TO-views.*

As is well-known, a view is just a play in which Opponent moves are justified by their predecessors:

Lemma 9. *A TO-play* $s = (n, f, \varphi)$ *is a TO-view iff for all odd* $i \in n$*,* $\varphi(i) = i - 1$*.*

The inclusion $\mathrm{i}_{TO}: \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$ induces in particular an adjunction

$$\begin{tikzcd} \widehat{\mathbb{P}_{A,B}} \arrow[r, bend left, "\Delta_{\mathrm{i}_{TO}}"] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{V}_{A,B}}, \arrow[l, bend left, "\Pi_{\mathrm{i}_{TO}}"] \end{tikzcd}$$

where Δ_F denotes restriction along F^{op} and $\prod_F$ denotes right Kan extension along F^{op}.

Definition 10. *We call* $\widehat{\mathbb{V}_{A,B}}$ *the category of* TO-behaviours*. The category* $\mathrm{Sh}(\mathbb{P}_{A,B})$ *of* innocent strategies *on* (A, B) *is the essential image of* $\prod_{\mathrm{i}_{TO}}$*.*

By construction, $\prod_{\mathrm{i}_{TO}}$ restricts to an equivalence $\mathrm{Sh}(\mathbb{P}_{A,B}) \simeq \widehat{\mathbb{V}_{A,B}}$.

2.2. **Pseudo double categories.** A *pseudo double category* $\mathbb{D}$ consists of a set $\mathrm{ob}(\mathbb{D})$ of *objects*, shared by a "horizontal" category $\mathbb{D}_h$ and a 'vertical' bicategory $\mathbb{D}_v$. Following Paré [28], $\mathbb{D}_h$, merely being a category, has standard notation (normal arrows, $\circ$ for composition, id for identities), while the bicategory $\mathbb{D}_v$ earns fancier notation ($\to\!\!\bullet\!\!\to$ for arrows, $\bullet$ for composition, $id^\bullet$ for identities). Moreover, for each "square" as on the left below, $\mathbb{D}$ comes equipped with a set of *cells* α, wich have vertical, resp. horizontal, domains and codomains, denoted by $\mathrm{dom}_v(\alpha)$, $\mathrm{cod}_v(\alpha)$, $\mathrm{dom}_h(\alpha)$, and $\mathrm{cod}_h(\alpha)$. We picture this as on the right below.

$$
\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[d, "u'"] \\
X \arrow[r, "h'"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "u"'] & Y' \arrow[d, "u'"] \\
X \arrow[r, "h'"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & Y,
\end{tikzcd}
$$

where $u = \mathrm{dom}_h(\alpha)$, $u' = \mathrm{cod}_h(\alpha)$, $h = \mathrm{dom}_v(\alpha)$, and $h' = \mathrm{cod}_v(\alpha)$. $\mathbb{D}$ is furthermore equipped with operations for composing cells: $\circ$ composes them along a common vertical morphism, $\bullet$ composes along horizontal morphisms. Both vertical compositions (of morphisms and cells) may only be associative up to coherent isomorphism. The full axiomatisation is given by Garner [12], and we here only mention the *interchange law*, which says that both ways of parsing the following diagram coincide:

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & X' \arrow[r, "k"] \arrow[d, "u'"] \arrow[dr, phantom, "\overset{\alpha'}{\Longrightarrow}"] & X'' \arrow[d, "u''"] \\
Y \arrow[r, "h'"] \arrow[d, "v"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & Y' \arrow[r, "k'"] \arrow[d, "v'"] \arrow[dr, phantom, "\overset{\alpha'}{\Longrightarrow}"] & Y'' \arrow[d, "v''"] \\
Z \arrow[r, "h''"'] & Z' \arrow[r, "k''"'] & Z'',
\end{tikzcd}
$$

i.e., $(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = (\beta' \bullet \alpha') \circ (\beta \bullet \alpha)$.

For any pseudo double category $\mathbb{D}$, we denote by $\mathbb{D}_H$ the category with vertical morphisms as objects and cells as morphisms, and by $\mathbb{D}_V$ the bicategory with horizontal morphisms as objects and cells as morphisms. Domain and codomain maps extend to functors $\mathrm{dom}_v, \mathrm{cod}_v : \mathbb{D}_H \to \mathbb{D}_h$ and $\mathrm{dom}_h, \mathrm{cod}_h : \mathbb{D}_V \to \mathbb{D}_v$. We will refer to dom_v and cod_v simply as dom and cod, reserving subscripts for dom_h and cod_h.

Here is one of the prime examples of a pseudo double category, on which our construction will be based.

Example 11. *Starting from any category $\mathcal{C}$ with pushouts, consider the pseudo double category $\mathsf{Cospan}(\mathcal{C})$ with*

- *$\mathcal{C}$ itself as horizontal category, i.e., $\mathsf{Cospan}(\mathcal{C})_h = \mathcal{C}$,*
- *as vertical morphisms $X \to Y$ all cospans $X \to U \leftarrow Y$, and*
- *as cells all commuting diagrams*

$$
\begin{tikzcd}
X \arrow[r, "k"] \arrow[d] & X' \arrow[d] \\
U \arrow[r, "l"] & U' \\
Y \arrow[r, "h"'] \arrow[u] & Y', \arrow[u]
\end{tikzcd}
\tag{2}
$$

with $\mathrm{dom}(k, l, h) = k$, $\mathrm{dom}_h(k, l, h) = (X \to U \leftarrow Y)$, etc.

Composition in $\mathsf{Cospan}(\mathcal{C})_v$ is defined by (some global choice of) pushout and composition in $\mathsf{Cospan}(\mathcal{C})_V$ follows by universal property of pushout.

Any signature S will comprise a base category $\mathbb{C}$, and $\mathbb{D}(\mathsf{S})$ will be a sub-pseudo double category of $\mathsf{Cospan}(\widehat{\mathbb{C}})$. Very roughly, $\mathbb{C}$ will be equipped with a notion of dimension, and S will consist of a selection of cospans

$$
Y \xrightarrow{s} M \xleftarrow{t} X \tag{3}
$$

in $\widehat{\mathbb{C}}$ viewed as morphisms $Y \to X$ in $\mathsf{Cospan}(\widehat{\mathbb{C}})_v$, where X and Y have dimension at most 1 and M may have arbitrary dimension. We will call these cospans *seeds.* The intuition is that presheaves of dimension ≤ 1 model *positions* in a game (they are essentially graphs), whilst higher-dimensional presheaves model the dynamics of the game. Thus the cospan (3) models a *play*, starting in position X and ending in position Y, and M models *how* the play evolves from X to Y. Up to some technicalities, $\mathbb{D}(\mathsf{S})$ is the smallest sub-pseudo double category of $\mathsf{Cospan}(\widehat{\mathbb{C}})$ whose objects are positions and whose vertical morphisms contain seeds.

2.3. **Pushouts, pullbacks and monos in presheaf categories.** In this final preliminary section, let us collect a few basic results about pushouts, pullbacks and monos in presheaf categories, which we will consider to be second nature in the sequel. The well-known pullback and pushout lemmas are not recalled, though widely used throughout. Similarly, let us merely recall that epis are stable under pullback, and that monos are stable under pushout in presheaf categories.

Let us start with an easy result about pullbacks along monos:

Lemma 12. *Any commuting square as below with j iso and m monic is a pullback:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & B \arrow[d, hook, "m"] \\ C \arrow[r] & D. \end{tikzcd}$$

Proof. A simple diagram chase. □

Our second result is specific to sets:

Lemma 13. *Any commuting square of the form below left is a pullback if each rectangle as below right is:*

$$\begin{tikzcd} \sum_{i \in I} A_i \arrow[r, "{[f_i]_{i \in I}}"] \arrow[d, "\sum_{i \in I} k_i"'] & A \arrow[d, "k"] \\ \sum_{i \in I} B_i \arrow[r, "{[g_i]_{i \in I}}"'] & B \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "k_i"'] & A \arrow[d, "k"] \\ B_i \arrow[r, "g_i"'] & B. \end{tikzcd}$$

Proof. Straightforward. □

Our third result is an instance of the other pullback lemma [29].

Lemma 14 (Another pullback lemma)**.** *In any presheaf category, for any commuting diagram as below with e epi, if the outer rectangle and the left square are pullbacks then so is the right square:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r, "e"'] & Y \arrow[r] & Z. \end{tikzcd}$$

Proof. An immediate consequence of [29, Theorem 1], given that epis are pointwise, hence strong, and stable under pullback in presheaf categories. □

Let us continue by recalling the adhesivity properties of presheaf categories [23].

Lemma 15. *In any presheaf category, any pushout along a mono is also a pullback. Explicitly, any pushout square*

$$\begin{tikzcd} A \arrow[r, hook, "m"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with m mono is also a pullback.

Lemma 16 (Adhesivity)**.** *In any presheaf category, for any commuting cube*

$$
\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr] \arrow[dd] & \\
& A \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
I' \arrow[rr, hook] \arrow[dr] & & B' \arrow[dr] & \\
& A' \arrow[rr] & & C' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

with the marked pullbacks, mono and pushout, the front faces are pullbacks iff the top face is a pushout.

Proof. By [23, Example 6 and Proposition 8 (iii)]. □

Let us finish with a similar-looking statement, which has in fact more to do with extensivity [5] of Set than with adhesivity.

Lemma 17. *In* Set*, for any commuting cube*

$$
\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr] \arrow[dd] & \\
& A \arrow[rr] \arrow[dd] & & C \arrow[dd, "f"] \arrow[ul, phantom, "\ulcorner", very near start] \\
I' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& A' \arrow[rr] & & C' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

with the marked pushouts and pullback,

- *if $I' \to B'$ is injective then the front square is a pullback, and*
- *if all arrows except perhaps f are injective, then f also is.*

Proof. The following proof is due to Paweł Sobociński (private communication). In Set, the map $m: I' \to B'$, being injective, may be written as a coproduct injection $m: I' \to I' + X'$. But, injective maps being stable under pullback and coproduct injections being stable under pushout, the whole cube may be written as

$$
\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & & I + X \arrow[dr] \arrow[dd, "g+k"] & \\
& A \arrow[rr] \arrow[dd, "h"'] & & A + X \arrow[dd, "f"] \arrow[ul, phantom, "\ulcorner", very near start] \\
I' \arrow[rr] \arrow[dr] & & I' + X' \arrow[dr] & \\
& A' \arrow[rr] & & A' + X', \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

with $f = h+k$. This in particular shows that injectivity of h and k entails injectivity of f. Let us now show that the front face is a pullback. Indeed, it is the pasting of both left-hand squares below:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & A + X \arrow[d, "A+k"] & X \arrow[l] \arrow[d, "k"] \\
A \arrow[r] \arrow[d, "h"'] & A + X' \arrow[d, "h+X'"] & X' \arrow[l] \arrow[d, equal] \\
A' \arrow[r] & A' + X' & X'. \arrow[l]
\end{tikzcd}
$$

All rows being coproduct injections, by extensivity all squares are pullbacks, hence so is the face of interest by the pullback lemma. □

Corollary 18. *For any commuting cube as in Lemma 17 in any presheaf category with the marked pushouts and pullback,*

- *if $I' \to B'$ is monic then the front square is a pullback, and*
- *if all arrows except perhaps f are monic, then f also is.*

Proof. Monos and pullbacks are pointwise in presheaf categories. □

3. Signatures for pseudo double categories

In this section, we contsruct our signature S_{HON} for HON games, defining the notion of signature more or less in parallel. We start by adapting the method used in our previous sheaf models to HON games, until we are able to state the definition of a signature. We then (abstractly) give the construction of the pseudo double category $\mathbb{D}(\mathsf{S})$ associated to any signature S. Finally, we motivate fibredness by showing how it occurs in the definition of relevant categories of plays.

3.1. **Towards a signature for HON games.**

3.1.1. *Method.* The method used in previous work to design games proceeds in four stages:

(i) Design a base category $\mathbb{C}_1$ over which finitely presentable presheaves will model positions in the game.

(ii) Select a collection of spans of monomorphisms modelling "typical" moves in the game. In each selected span $Y \xleftarrow{d} Z \xrightarrow{c} X$,
- X denotes the initial position of the move,
- Y denotes the final position of the move,
- Z denotes the part of the position which remains unchanged during the move.

This is in line with the double pushout approach to graph rewriting [10].

(iii) The next step is characteristic of our approach, and allows us to give a causal representation of plays not as spans but rather as cospans (on a richer base category). The obtained representation is significantly simpler [6] than analogous, span-based representations of rewrite sequences [3]. It proceeds by augmenting $\mathbb{C}_1$ to a category $\mathbb{C}$ over which finitely presentable presheaves will also model plays. This goes by adding one new object μ_S for each selected span $S = (Y \xleftarrow{s} Z \xrightarrow{t} X)$, in a way that

$$\begin{tikzcd} Z \arrow[r, hook, "d"] \arrow[d, hook, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "s"] \\ X \arrow[r, hook, "t"'] & \mu_S \end{tikzcd}$$

is a pullback (with the marked monos). The resulting cospans $Y \xrightarrow{s} \mu_S \xleftarrow{t} X$ are called *seeds* of the game. In fact, this description of seeds is slightly naive, as it implies that any two seeds are independent from each other, a simplification that we cannot afford if we aim at fibredness. So we should also describe morphisms between the new objects, and select our spans in a compatible way, which essentially means that they should induce a functor $\mathsf{S}: \mathbb{C}_{\geq 2} \to \mathsf{Cospan}(\widehat{\mathbb{C}})_H$, where $\mathbb{C}_{\geq 2}$ is the full subcategory of $\mathbb{C}$ spanning the new objects.

(iv) In the last stage, roughly (see Section 3.3 for details), we construct the smallest sub-pseudo double category of $\mathsf{Cospan}(\widehat{\mathbb{C}})$

- whose objects are positions,
- whose vertical morphisms include all identities and seeds, and
- which contains all pushouts of the form

$$\begin{tikzcd} id^\bullet_Z \arrow[r, "id^\bullet_h"] \arrow[d] & id^\bullet_{Z'} \arrow[d] \\ \mathsf{S}(\mu_S) \arrow[r] & M \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

for all selected spans $S = (Y \leftarrow Z \rightarrow X)$ and morphism $h{:}\, Z \rightarrow \mathbb{Z}'$ with Z' a position.

The last point intuitively says that moves (modelled by seeds) may occur in context.

We will handle the first three stages here, which will in fact yield our example signature S_{HON}. The last step will result from the general process associating a pseudo double category $\mathbb{D}(\mathsf{S})$ to each signature S.

3.1.2. *Sequent calculus and informal description of the game.* A preliminary step is to organise standard arena games into a sequent calculus, which will then guide us through steps *(i)–(iii)*. This is close in spirit to Melliès's work [26] – though the latter takes place in a linear setting. The idea is to understand an arena $A = \sum_i m_i.A_i$ as a logical formula much like $\bigwedge_i \neg A_i$, and to consider the straightforward focalised [2] sequent calculus on this.

A *sequent* is a list of arenas, possibly with a distinguished formula, denoted by $A_1, \ldots, A_n \vdash$, resp. $A_1, \ldots, A_n \vdash A$, and our sequent calculus has the following inference rules:

$$\textsc{Right}\ \frac{\ldots \quad \Gamma, A_i \vdash \quad \ldots \quad (\forall i \in I)}{\Gamma \vdash \sum_{i \in I} m_i.A_i} \qquad \textsc{Left}\ \frac{\Gamma, \sum_i m_i.A_i, \Delta \vdash A_i}{\Gamma, \sum_i m_i.A_i, \Delta \vdash}$$

$$\textsc{Cut}\ \frac{\Gamma \vdash A \qquad \Delta, A, \Theta \vdash B}{\Delta, \Gamma, \Theta \vdash B} \qquad \textsc{Cut'}\ \frac{\Gamma \vdash A \qquad \Delta, A, \Theta \vdash}{\Delta, \Gamma, \Theta \vdash}\ .$$

This sequent calculus comes with the following cut elimination rule (ommitting contexts):

$$\frac{\dfrac{\pi}{\vdash A} \qquad \dfrac{\dfrac{\pi'}{A \vdash A_i}}{A \vdash}}{\vdash} \quad \rightsquigarrow \quad \frac{\dfrac{\dfrac{\pi}{\vdash A} \qquad \dfrac{\pi'}{A \vdash A_i}}{\vdash A_i} \qquad \dfrac{\pi_i}{A_i \vdash}}{\vdash}$$

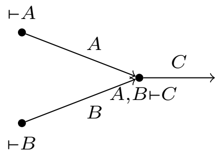
where $A = \sum_i m_i.A_i$ and π is

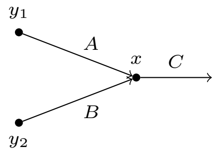
$$\frac{\ldots \quad \dfrac{\pi_i}{A_i \vdash} \quad \ldots \quad (\forall i \in I)}{\vdash A}\ .$$

Let us now informally describe the game based on this sequent calculus. A position will consist of a finite family of sequents, connected as prescribed by the Cut and Cut′ rules – that is, by linking the conclusion of some sequent to some hypothesis of another. Each sequent will represent a player. E.g., giving a strategy on such a position will consist of giving one for each sequent. In particular, the game is definitely multi-party.

Example 19. *The* Cut *derivation below left yields the position below center.*

$$\dfrac{\vdash B \qquad \dfrac{\vdash A \qquad A, B \vdash C}{B \vdash C}}{\vdash C}$$





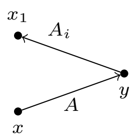
.

The sequent at any vertex x is determined by the arenas and directions of adjacent edges. It may not be obvious that the order of hypotheses is: this follows from the notion of graph that we use, see below. Thus, we omit to label vertices with sequents and only keep labels on edges. It is often useful to name vertices, so that the involved position might be drawn as below right.

In fact, we will allow more general connectivity than the simple trees prescribed by the Cut rules, but let us ignore this for the moment.

Let us now describe moves in the game. Typically, a move will involve two sequents connected by an edge, and will correspond to the relevant instance of the cut elimination rule above. So let us consider the simplest generic case: the position $\overset{x}{\bullet} \xrightarrow{A} \overset{y}{\bullet}$, omitting contexts for the moment. In this position, the sequent for x is $\vdash A$ and the one for y is $A \vdash$. Let $A = \sum_i m_i.A_i$. The idea is that the position is described by the bottom part of the redex of the cut elimination rule, π and π' intuitively corresponding to some strategies for both players. If both strategies accept the move, they will end up in a position described by the bottom part of the reactum of the cut elimination rule, namely:

(4)



.

A subtle point here is that after the move, the player holding the sequent $\vdash A$ should behave the same as x did before the move, a requirement similar to so-called *uniformity* in game semantics. Thus, x should not even be aware that the move has been played. Technically, this will be modelled by enforcing the fact that x has a so-called *identity* view of the move, and then by restricting attention to innocent strategies.

Finally, assume that in the initial position, x had some hypothesis, say B. Then after the move, x and x_1 both should have access to the same B. It thus appears that we need sequents with multiple outputs or, rather, sequents with *shared* outputs.

3.1.3. *Stage* (i)*: positions.* Let us now formalise this. All involved data will be represented as presheaves on a certain base category, say $\mathbb{L}$. Let us start with $\mathbb{L}_1$, the part of $\mathbb{L}$ which only concerns positions:

Definition 20. *Let $\mathbb{L}_1$ have*

- *an object for each arena;*
- *an object for each sequent;*
- *morphisms $s_i: A_i \to S$ for each sequent $S = (A_1, \ldots, A_n \vdash)$;*
- *morphisms $s_i: A_i \to S$ (for all $i \in n$) and $t: A \to S$ for each sequent $S = (A_1, \ldots, A_n \vdash A)$.*

Example 21. *The (informal) position of Example 19 is modelled as the presheaf X with $X(A) = \{a\}$, $X(B) = \{b\}$, $X(C) = \{c\}$, $X(\vdash A) = \{y_1\}$, $X(\vdash B) = \{y_2\}$, $X(A, B \vdash C) = \{x\}$, and empty otherwise, whose action on morphisms is determined by $y_1 \cdot t = a$, $y_2 \cdot t = b$, $x \cdot t = c$, $x \cdot s_1 = a$, $x \cdot s_2 = b$.*

3.1.4. *Stage* (ii)*: selecting spans.* Let us now select the spans that will model moves. For all arenas $A = \sum_{i \in I} m_i.A_i$ and $i \in I$, we will have a span β corresponding to the move sketched in Example 19. But in order for the obtained pseudo double category to be fibred (which basically means that there is a canonical way to restrict any play to any sub-position of its initial position), we need to introduce one additional move for each player in the initial position. This makes two moves: the one corresponding to the RIGHT rule will be denoted by Λ, the one corresponding to LEFT being denoted by @. As announced above, we should enforce the fact that the player x in (4) has an identity view of β and therefore of Λ.

Given this, let us now select spans for the Λ, @, and β moves.

For all sequents $S = (B_1, \ldots, B_m \vdash A)$ and $q \in \sqrt{A}$, consider the span $\mathcal{S}^{\Lambda}_{S,q}$ =

$$Y^{\Lambda}_{S,q} \xleftarrow{d^{\Lambda}_{S,q}} Z^{\Lambda}_{S,q} \xrightarrow{c^{\Lambda}_{S,q}} X^{\Lambda}_{S,q},$$

where

- $X^{\Lambda}_{S,q} = S$ (by Section 1, this is implicitly y_S),
- $Y^{\Lambda}_{S,q}$, which we will henceforth denote by $S|(B_1, \ldots, B_m, A \cdot q \vdash)$, is the pushout

$$\begin{tikzcd} \sum_{j \in m} B_j \arrow[r, "{[s_j]_{j \in m}}"] \arrow[d, "{[s_j]_{j \in m}}"'] & (B_1, \ldots, B_m, A \cdot q \vdash) \arrow[d, "\mathrm{inj_r}"] \\ S \arrow[r, "\mathrm{inj_l}"'] & (S|(B_1, \ldots, B_m, A \cdot q \vdash)) \end{tikzcd}$$

- and $Z^{\Lambda}_{S,q} = S$,

with the obvious morphisms. The possibly unclear point is the presence of $\sum_j B_j$ in the pushout: this is due to the necessity for all hypotheses of x to be shared with x_1 in (4).

For all $S' = (A_1, \ldots, A_n \vdash)$, $i' \in n$, and $q \in \sqrt{A_i}$, consider the span $\mathcal{S}^{@}_{S',i,q}$ =

$$Y^{@}_{S',i,q} \xleftarrow{d^{@}_{S',i,q}} Z^{@}_{S',i,q} \xrightarrow{c^{@}_{S',i,q}} X^{@}_{S',i,q},$$

where

- $X^{@}_{S',i,q} = S'$,
- $Y^{@}_{S',i,q} = (A_1, \ldots, A_n \vdash A_i \cdot q)$
- and $Z^{@}_{S',i,q} = \sum_{j \in n} A_j$.

Finally, let us treat β.

Notation 22. *For all sequents $S = (\Gamma \vdash A)$, $S' = (A_1, \ldots, A_n \vdash)$, and $i \in n$ such that $A_i = A$, let us denote by $S \rhd_i S'$ the pushout*

$$\begin{tikzcd} A \arrow[r, "s_i"] \arrow[d, "t"'] & S' \arrow[d, "\mathrm{inj_r}"] \\ S \arrow[r, "\mathrm{inj_l}"'] & (S \rhd_i S') \end{tikzcd}$$

(we will drop the i subscript when unambiguous).

For all sequents $S = (B_1, \ldots, B_m \vdash A)$ and $S' = (A_1, \ldots, A_n \vdash)$ with $A_i = A$ and $q \in \sqrt{A}$, consider the span $\mathcal{S}^{\beta}_{S,S',i,q}$ =

$$Y^{\beta}_{S,S',i,q} \xleftarrow{d^{\beta}_{S,S',i,q}} Z^{\beta}_{S,S',i,q} \xrightarrow{c^{\beta}_{S,S',i,q}} X^{\beta}_{S,S',i,q},$$

where

- $X^{\beta}_{S,S',i,q} = (S \rhd S')$,
- $Y^{\beta}_{S,S',i,q}$ denotes the pushout

$$\begin{tikzcd}
A + (A \cdot q) \arrow[rr, "{[s_i, t]}"] \arrow[d, "{[\mathrm{inj_l} \circ t, \mathrm{inj_r} \circ s_{m+1}]}"'] & & (A_1, \ldots, A_n \vdash A \cdot q) \arrow[d, "\mathrm{inj_r}"] \\
(S|(B_1, \ldots, B_m, A \cdot q \vdash)) \arrow[rr, "\mathrm{inj_l}"'] & & Y^\beta_{S,S',i,q}
\end{tikzcd}$$

- and $Z^\beta_{S,S',i,q} = (S + \sum_{j \in n} A_j)$,

with the obvious morphisms.

3.1.5. *Stage* (iii)*: augmenting the base category.* At last, we augment our base category $\mathbb{L}_1$ with new objects and morphisms that model moves. We could here appeal to cocomma categories, as sketched in [6], which would lead to the definition below, except it would lack the morphisms λ and @.

Definition 23. *Let* $\mathbb{L}$ *consist of* $\mathbb{L}_1$*, plus:*

- *For all sequents* $S = (B_1, \ldots, B_n \vdash A)$ *and* $q \in \sqrt{A}$*, an object* $\Lambda_{S,q}$ *with maps* $S \xrightarrow{t} \Lambda_{S,q} \xleftarrow{s} (B_1, \ldots, B_n, A \cdot q \vdash)$ *such that* $t \circ s_j = s \circ s_j$ *for all* $j \in n$*;*
- *For all sequents* $S = (A_1, \ldots, A_n \vdash)$*,* $i \in n$*, and* $q \in \sqrt{A_i}$*, an object* $@_{S,i,q}$ *with maps* $S \xrightarrow{t} @_{S,i,q} \xleftarrow{s} (A_1, \ldots, A_n \vdash A_i \cdot q)$ *such that* $t \circ s_i = s \circ s_i$ *for all* $i \in n$*;*
- *For all sequents* $S = (B_1, \ldots, B_m \vdash A)$ *and* $S' = (A_1, \ldots, A_n \vdash)$*, with* $i \in n$ *such that* $A_i = A$ *and* $q \in \sqrt{A}$*, an object* $\beta_{S,S',i,q}$ *with maps* $\Lambda_{S,q} \xrightarrow{\lambda} \beta_{S,S',i,q} \xleftarrow{@} @_{S',i,q}$ *such that*

$$\lambda \circ t \circ t = @ \circ t \circ s_i \qquad \text{and} \qquad \lambda \circ s \circ s_{m+1} = @ \circ s \circ t.$$

The pseudo double category describing HON games will be a sub-pseudo double category of $\mathsf{Cospan}(\widehat{\mathbb{L}})$, entirely determined by the choice of a functor from $\mathbb{L}_{\geq 2}$ to $\mathsf{Cospan}(\widehat{\mathbb{L}})_H$. Intuitively, this functor chooses for all new objects μ of $\mathbb{L}$ a cospan which will be thought of as the basic play consisting of just μ. This choice of cospan should of course be compatible with the choice of spans made above, as mentioned in the informal description of the process in 3.1.1.

In fact, for each valid tuple (S, S', i, q), $\mathbb{L}_{\geq 2}$ locally looks like the poset

$$\begin{tikzcd}
& \beta_{S,S',i,q} & \\
\Lambda_{S,q} \arrow[ur, "\lambda"] & & @_{S',i,q} \arrow[ul, "@"']
\end{tikzcd}$$

viewed as a category. We define our functor $\mathbb{L}_{\geq 2} \to \mathsf{Cospan}(\widehat{\mathbb{L}})_H$ to map this to

$$\begin{tikzcd}
S \mid (B_1, \ldots, B_m, A \cdot q \vdash) \arrow[r, "\mathrm{inj_l}"] \arrow[d, hook] & Y^\beta_{S,S',i,q} \arrow[d, hook] & (A_1, \ldots, A_n \vdash A \cdot q) \arrow[l, "\mathrm{inj_r}"'] \arrow[d, hook] \\
\Lambda_{S,q} \arrow[r, "\lambda"] & \beta_{S,S',i,q} & @_{S',i,q} \arrow[l, "@"'] \\
S \arrow[r, "\mathrm{inj_l}"'] \arrow[u, hook] & S \rhd_i S' \arrow[u, hook] & S'. \arrow[l, "\mathrm{inj_r}"] \arrow[u, hook]
\end{tikzcd} \tag{5}$$

Definition 24. *Let* $\mathsf{S}_{HON} : \mathbb{L}_{\geq 2} \to \mathsf{Cospan}(\widehat{\mathbb{L}})_H$ *denote the obtained functor.*

We now need to construct our pseudo double category based on this. Before proceeding, as we intend to provide a generic construction, we reflect a bit on the properties of our functor S_{HON}, which leads us to our notion of signature.

3.2. **Signatures.** Our first observation is that the category $\mathbb{L}$ enjoys a natural notion of dimension: each A has dimension 0, each sequent has dimension 1, each $\Lambda_{S,q}$ and $@_{S',i,q}$ have dimension 2, and each $\beta_{S,S',i,q}$ has dimension 3. In particular, $\mathbb{L}$ forms a *direct* category in the sense of Garner [11], i.e., it comes equipped with a functor to the ordinal ω viewed as a category, which reflects identities. Presheaves X themselves inherit a (possibly infinite) dimension: the least n such that X is empty above dimension n. The dimension of any representable is thus that of the underlying object.

Second, let us make a few additional observations on our functor $\mathsf{S}_{HON}: \mathbb{L}_{\geq 2} \to \mathsf{Cospan}(\widehat{\mathbb{L}}_f)_H$:

(a) the middle object of each $\mathsf{S}_{HON}(\mu)$ is y_μ;
(b) both legs of all selected cospans are monic;
(c) all morphisms between those cospans have monic components;
(d) for all such morphisms, both the bottom and top squares are pullbacks;
(e) finally, all initial positions X are *tight*, in the sense that all channels $a \in X(A)$ are in the image of some $X(s_i)$ or $X(t)$.

So a first, naive notion of signature could consist of a direct category $\mathbb{C}$, equipped with a functor from $\mathbb{C}_{|\geq 2}$ to $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)_H$ satisfying *(a)–(e)*. However, some of the examples we have in mind require a bit more generality, so in our abstract definition we relax things a bit. Let us briefly explain why we need to relax the definition.

In our model of the π-calculus [7], there is a morphism between seeds whose bottom square is not a pullback; and there are natural, though unpublished examples in which the top square is not a pullback either. Similarly, in [7], we need the generated pseudo double category to accomodate morphisms of cospans whose components are non-injective. Let us thus generalise our tentative definition just enough to accept these examples. In short, we pass from injective maps to maps which are injective except perhaps on channels, and find an analogous generalisation for bottom squares being pullbacks. We in fact completely drop the requirement about top squares.

Let us fix any small, direct category $\mathbb{C}$ for the rest of this section. By analogy with $\mathbb{L}$, we think of objects of dimension 1 as players in a game, which may communicate with each other through objects of dimension 0. Objects of dimensions > 1 are thought of as moves in the game. Accordingly, we use the following terminology:

Terminology 25. *The* dimension *of any object of* $\mathbb{C}$ *is its image in* ω. *A* channel *is an object of dimension* 0*; a* player *is an object of dimension* 1*; a* move *is an object of dimension* > 1.

Definition 26. *Let a natural transformation of presheaves over* $\mathbb{C}$ *be* 1D-injective *when all its components of dimensions* > 0 *are injective.*

A square in $\widehat{\mathbb{C}}$ *is a* 1D-pullback *when it is a pullback in all dimensions* > 0.

Notation 27. *We mark 1D-pullbacks with a dotted little square, as below left*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

whilst dashed little squares as on the right merely indicate a pullback in dimension 1.

Definition 28. *Let* $\mathbb{D}^0(\mathbb{C})$ *denote the sub-pseudo double category of* $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$

- *whose horizontal category* $\mathbb{D}^0(\mathbb{C})_h$ *is the subcategory of* $\widehat{\mathbb{C}}_f$ *consisting of* positions*, i.e., finitely presentable presheaves of dimension* ≤ 1*, and 1D-injective morphisms between them,*

- *whose vertical morphisms are cospans with monic legs, and*
- *whose cells are those of* $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$ *with 1D-injective components and 1D-pullback bottom squares.*

Terminology 29. *For any vertical* $u: Y \multimap X$ *in* $\mathbb{D}^0(\mathbb{C})_v$*,* X *and* Y *are respectively called the* initial *and* final *positions of* u*.*

Definition 28 only makes sense because:

Proposition 30. $\mathbb{D}^0(\mathbb{C})$ *forms a sub-pseudo double category of* $\mathsf{Cospan}(\widehat{\mathbb{C}}_f)$*.*

This relies on the following direct corollary of Lemma 17:

Corollary 31. *In* $\widehat{\mathbb{C}}$*, for any commuting cube*

$$\begin{tikzcd} I \arrow[rr] \arrow[dr] \arrow[dd] & & B \arrow[dr] \arrow[dd] & \\ & A \arrow[rr] \arrow[dd] & & C \arrow[dd, "f"] \\ I' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\ & A' \arrow[rr] & & C', \end{tikzcd}$$

with the marked pushouts and 1D-pullback,

- *if* $I' \to B'$ *is 1D-injective then the front square is a 1D-pullback, and*
- *if all arrows except perhaps* f *are 1D-injective, then* f *also is.*

Proof. By pointwise application of Lemma 17. □

Proof of Proposition 30. The only non-trivial bit lies in showing that a vertical composite of componentwise 1D-injective cells with 1D-pullback bottom squares again has 1D-injective components and 1D-pullback bottom square. This is a simple consequence of Corollary 31 and the pullback lemma. □

Let us finally give a definition of tightness which generalises the one given above for $\mathbb{L}$ – though the presentation differs.

Definition 32. *For all presheaves* U *in* $\widehat{\mathbb{C}}$*, let us denote by* $\mathrm{pl}(U)$ *the set of* players *of* U*, i.e., pairs* (d, x) *for all morphisms* $x: d \to U$*, where* d *is any representable of dimension* 1*. Let* $\mathrm{Pl}(U)$ *denote the corresponding coproduct* $\sum_{(d,x)\in \mathrm{pl}(U)} d$ *of representables.*

A position X *is* tight *iff the canonical morphism* $\mathrm{Pl}(X) \to X$ *is epi.*

Definition 33. *A* signature *consists of a small, direct category* $\mathbb{C}$*, together with a functor* $\mathsf{S}: \mathbb{C}_{\geq 2} \to \mathbb{D}^0(\mathbb{C})_H$ *making the following square commute*

(6)
$$\begin{tikzcd} \mathbb{C}_{\geq 2} \arrow[r, "\mathsf{S}"] \arrow[d, hook] & \mathbb{D}^0(\mathbb{C})_H \arrow[d, "\mathsf{m}"] \\ \mathbb{C} \arrow[r, hook, "\mathsf{y}"'] & \widehat{\mathbb{C}}_f, \end{tikzcd}$$

where m *denotes the middle projection functor. We further require that for all* $\mu \in \mathbb{C}_{\geq 2}$*, the initial position of* $\mathsf{S}(\mu)$ *is tight.*

Definition 34. *Cospans in the image of* S *are called the* seeds *of* S*.*

Letting $X, Y, Z, \ldots$ range over positions, we get that any signature S maps any move M to some cospan $Y \to M \leftarrow X$ which determines its initial and final positions.

Example 35. *The functor S_{HON} of Definition 24 is a signature. Indeed, all morphisms are evidently monic and both bottom squares are straightforwardly pullbacks, hence 1D-pullbacks. Furthermore, all initial positions are clearly tight.*

Here is an immediate, useful consequence of the definition:

Lemma 36. *The functor S underlying any signature is fully faithful.*

Proof. Faithfulness is trivial and fullness follows from monicity of legs of the involved cospans. □

3.3. **The construction.** We now define and give an explicit description of the pseudo double category $\mathbb{D}(\mathsf{S})$ associated to any signature S.

Let us start with the following observation:

Proposition 37. *for any pushout square of the form*

$$\begin{tikzcd} id^\bullet_{Z_0} \arrow[r, "id^\bullet_h"] \arrow[d, "k"'] & id^\bullet_Z \arrow[d] \\ \mathsf{S}(\mu) \arrow[r] & M \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{7}$$

in $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$, if $h \in \mathbb{D}^0(\mathbb{C})_h(Z_0, Z)$ and $k \in \mathbb{D}^0(\mathbb{C})_H(id^\bullet_{Z_0}, \mathsf{S}(\mu))$, i.e., h is 1D-injective and k has 1D-injective components and 1D-pullback bottom square, then the whole square in fact lies in $\mathbb{D}^0(\mathbb{C})$.

Proof. Indeed, given h and $id^\bullet_{Z_0} \to \mathsf{S}(\mu)$ as above, the pushout M always exists in $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$. It is computed by taking pushouts levelwise, as in

$$\begin{tikzcd} & Y_0 \arrow[rr] \arrow[d] & & Y \arrow[d, dashed] \\ & \mu \arrow[rr] & & M \\ Z_0 \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & Z \arrow[uur] \arrow[ur] \arrow[dr] & \\ & X_0 \arrow[uu] \arrow[rr] & & X, \arrow[uu, dashed] \end{tikzcd} \tag{8}$$

where $\mathsf{S}(\mu) = (Y_0 \to \mu \leftarrow X_0)$ and the dashed arrows are obtained by universal property of pushout. Now, monos are stable under pushouts in Set and colimits are pointwise in presheaf categories, so 1D-injectivity of all components follows from 1D-injectivity of all involved morphisms. Finally, both bottom squares are pullbacks, hence 1D-pullbacks as desired, by Lemmas 12 and 16. □

Definition 38. *A* move *is any cospan M obtained as some pushout of the form* (7).

Definition 39. *The pseudo double category $\mathbb{D}(\mathsf{S})$ associated to any signature S is the smallest sub-pseudo double category of $\mathbb{D}^0(\mathbb{C})$ such that*

- *$\mathbb{D}(\mathsf{S})_h$ is $\mathbb{D}^0(\mathbb{C})_h$;*
- *$\mathbb{D}(\mathsf{S})_H$ is replete and contains all moves;*
- *$\mathbb{D}(\mathsf{S})$ is* locally full, *i.e., if a cell of $\mathbb{D}^0(\mathbb{C})$ has its perimeter in $\mathbb{D}(\mathsf{S})$, then it is in $\mathbb{D}(\mathsf{S})$.*

Remark 4. *By the proposition, saying that $\mathbb{D}(\mathsf{S})$ contains all moves entails that it contains all associated pushout squares.*

That $\mathbb{D}(\mathsf{S})$ is well-defined is easy: it is the intersection of all sub-pseudo double categories of $\mathbb{D}^0(\mathbb{C})$ that verify all three points above, and $\mathbb{D}^0(\mathbb{C})$ is obviously one such pseudo double category, so we are taking the intersection of a non-empty family.

It is still useful to give a concrete description of $\mathbb{D}(\mathsf{S})$. First, its horizontal category is just $(\mathbb{D}^0(\mathbb{C}))_h$. Regarding vertical morphisms, $\mathbb{D}(\mathsf{S})$ must contain all moves, and since it should be stable under vertical composition it must also contain all finite composites of moves. By repleteness, it should also contain all vertical morphisms isomorphic to such vertical composites. We thus define:

Definition 40. *A* play *is any vertical morphism isomorphic to some vertical composite of moves.*

Proposition 41. $\mathbb{D}(\mathsf{S})$ *is precisely the locally full sub-pseudo double category of* $\mathbb{D}^0(\mathbb{C})$ *obtained by restricting vertical morphisms to plays.*

Proof. By construction, it is enough to show that the given data forms a sub-pseudo double category of $\mathsf{Cospan}(\widehat{\mathbb{C}})$, which is easy. ☐

3.4. **Fibredness and categories of plays.** For a given pseudo double category $\mathbb{D}$, the categories of plays studied in previous work [19, 7] come in several flavours. A first variant is based on the following category:

Definition 42. *Let* $\mathbb{E}$ *denote the category*

- *whose objects are vertical morphisms of* $\mathbb{D}$,
- *and whose morphisms* $u \to u'$ *are pairs* (w, α) *as below left, considered equivalent up to the equivalence relation generated by equating* (w, α) *with* $(w', \alpha \circ (u \bullet \gamma))$, *for all cells* γ *as below right:*

$$(9)\qquad \begin{tikzcd} T \arrow[r, "s"] \arrow[d, "w"'] & Z' \arrow[dd, "u'"] \\ Z \arrow[r, phantom, "\xRightarrow{\alpha}"] \arrow[d, "u"'] & \\ Y \arrow[r, "r"] & Y' \end{tikzcd} \qquad\qquad \begin{tikzcd} T' \arrow[r] \arrow[dr, "w'"'] \arrow[r, phantom, shift right=3, "\xRightarrow{\gamma}"] & T \arrow[r, "s"] \arrow[d, "w"] & Z' \arrow[dd, "u'"] \\ & Z \arrow[r, phantom, "\xRightarrow{\alpha}"] \arrow[d, "u"'] & \\ & Y \arrow[r, "r"'] & Y'. \end{tikzcd}$$

Notation 43. *We denote the involved equivalence relation by* $\sim$. *Furthermore, in principle,* $\mathbb{E}$ *depends on* $\mathbb{D}$, *which should appear in the notation. For readability, we will rely on context to disambiguate.*

In order to define composition in this category, one needs to consider all diagrams of shape the solid part of

$$\begin{tikzcd} Z'' \arrow[r, dashed] \arrow[d, dashed, "w''"'] \arrow[r, phantom, shift right=4, "\xRightarrow{\gamma}"] & Z' \arrow[r, "s'"] \arrow[d, "w'"'] & Y'' \arrow[ddd, "u''"] \\ Z \arrow[r, "s"] \arrow[d, "w"'] & Y' \arrow[dd, "u'"] \arrow[r, phantom, "\xRightarrow{\beta}"] & \\ Y \arrow[r, phantom, "\xRightarrow{\alpha}"] \arrow[d, "u"'] & & \\ X \arrow[r, "r"'] & X' \arrow[r, "r'"'] & X''. \end{tikzcd}$$

Fibredness then comes in by requiring the existence of a cell γ as shown, which is canonical in a certain sense. This allows us to define the composite of (w, α) and (w', β) as the equivalence class of $(w \bullet w'', \beta \circ (\alpha \bullet \gamma))$.

To formally state fibredness, let us recall [21] that for any functor $p\colon \mathcal{E} \to \mathcal{B}$, a morphism $r\colon E' \to E$ in $\mathcal{E}$ is *cartesian* when, as below, for all $t\colon E'' \to E$ and $k\colon p(E'') \to p(E')$ such that $p(r) \circ k = p(t)$, there exists a unique $s\colon E'' \to E'$ such that $p(s) = k$ and $r \circ s = t$:

$$\begin{tikzcd}
E'' \arrow[dr, dashed, "s"'] \arrow[drr, bend left, "t"] \arrow[dd] & & \\
& E' \arrow[r, "r"'] \arrow[dd] & E \arrow[dd] \\
p(E'') \arrow[dr, "k"'] \arrow[drr, bend left, "p(t)"] & & \\
& p(E') \arrow[r, "p(r)"'] & p(E).
\end{tikzcd}$$

Definition 44. *A functor $p: \mathcal{E} \to \mathcal{B}$ is a* fibration *iff for all $E \in \mathcal{E}$, any $h: B' \to p(E)$ has a cartesian lifting, i.e., a cartesian antecedent by p.*

Here is the long awaited fibredness property:

Definition 45. *A pseudo double category $\mathbb{D}$ is* fibred *iff the functor* $\mathrm{cod}: \mathbb{D}_H \to \mathbb{D}_h$ *is a fibration.*

Fibredness is related to Grandis and Paré's double categorical Kan extensions [15] and to Shulman's *framed* bicategories [30].

Proposition 46. *If $\mathbb{D}$ is fibred, then $\mathbb{E}$ is indeed a category.*

The category of plays $\mathbb{E}_X$ over any position X used in [19, 7] is then obtained as comma categories of $\mathbb{E}$ over the functor $\ulcorner X \urcorner : 1 \to \mathbb{D}_h$ picking X. For our study of TO-views and plays, we will use another variant:

Definition 47. *Let $\mathbb{E}(X)$ denote the fibre of $\mathbb{E}$ over X.*

Explicitly, objects of $\mathbb{E}(X)$ are plays $u: Y \multimap X$, and morphisms are those of $\mathbb{E}$, as on the left in (9), which have id_X as their lower border.

Our next goal is now to prove that $\mathbb{D}(\mathsf{S}_{HON})$ is indeed fibred, which we will then apply to compare the obtained categories $\mathbb{E}(X)$ to TO-plays. As announced, we will proceed abstractly.

4. Fibredness

In this section, we study the fibredness property abstractly. In order to do so, we first need to recall some facts about factorisation systems, which will be a crucial tool in our investigation. We then recast the definitions of 1D-injectivity and 1D-pullbacks in terms closer to the defining properties of injective maps and pullbacks. We then give a necessary and sufficient condition for $\mathbb{D}(\mathsf{S})$ to be fibred. However, this condition is not very useful in practice, so, in the last part, we give a sufficient condition for $\mathbb{D}(\mathsf{S})$ to be fibred that is easier to verify.

4.1. **Cofibrantly generated factorisation systems and fibredness.** Our main tool to prove that the pseudo double category $\mathbb{D}(\mathsf{S})$ generated by a signature S is fibred will be *cofibrantly generated factorisation systems*: let us recall their definition. In any category $\mathcal{C}$, we say that $l: A \to C$ is *left orthogonal* to $r: B \to D$, or equivalently that r is *right orthogonal* to l, iff for all commuting squares as the solid part of

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "l"'] & B \arrow[d, "r"] \\
C \arrow[r, "v"'] \arrow[ur, dashed, "d"] & D,
\end{tikzcd}$$

there exists a unique d as shown making both triangles commute.

Notation 48. *We denote by $l \perp r$ the existence of a unique such d for all u and v, and extend the notation to sets of arrows by writing $\mathcal{L} \perp \mathcal{R}$ when $l \perp r$ for all $l \in \mathcal{L}$*

and $r \in \mathcal{R}$*. Similarly,* $\mathcal{L}^{\perp}$ *denotes the class of all arrows that are right orthogonal to all arrows of* $\mathcal{L}$*, and symmetrically for* ${}^{\perp}\mathcal{R}$*.*

Definition 49. *A* factorisation system *on a category* $\mathcal{C}$ *consists of two classes of maps* $\mathcal{L}$ *and* $\mathcal{R}$ *such that* $\mathcal{L} = {}^{\perp}\mathcal{R}$*,* $\mathcal{L}^{\perp} = \mathcal{R}$*, and any morphism* $f: C \to D$ *factors as* $C \xrightarrow{l_f} A_f \xrightarrow{r_f} D$ *with* $l_f \in \mathcal{L}$ *and* $r_f \in \mathcal{R}$*.*

Example 50. *The first example of a factorisation system is given by the classes* Epi *and* Mono*, respectively of surjections and injections, in sets. This extends to presheaf categories: for any small category* $\mathbb{C}$*, epi and monic natural transformations form a factorisation system on* $\widehat{\mathbb{C}}$*, which we also denote by* (Epi, Mono)*.*

Cofibrant generation refers to the fact that $\mathcal{L}$ and $\mathcal{R}$ are defined from some generating set J, merely by the lifting property: $\mathcal{R} = J^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$. The point here is that J is a set, rather than a class. In fact, in many useful cases, it is even a rather small set. In our case, it will be bounded by the cardinality of $\mathbb{C}$. Though it is not trivial – this uses the famous "small object" argument, we have:

Theorem 51 (Bousfield [4])**.** *For any set* J *of maps in any cocomplete category* $\mathbb{C}$*,* $({}^{\perp}(J^{\perp}), J^{\perp})$ *forms a factorisation system.*

Example 52. *The* (Epi, Mono) *factorisation system on* Set *is cofibrantly generated by the singleton* $\{2 \to 1\}$*. For any* $\mathbb{C}$*, the* (Epi, Mono) *factorisation system on* $\widehat{\mathbb{C}}$ *is cofibrantly generated by the set of all maps* $[\mathsf{y}_{id_c}, \mathsf{y}_{id_c}]: \mathsf{y}_c + \mathsf{y}_c \to \mathsf{y}_c$*, for* $c \in \mathrm{ob}(\mathbb{C})$*.*

Let us now explain the core idea of the proof of fibredness. In the setting of Example 11, any factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ yields a fibred sub-pseudo double category of $\mathsf{Cospan}(\mathcal{C})$. Indeed, recall the following well-known result [4, Lemma 2.4]:

Lemma 53. *For any factorisation system* $(\mathcal{L}, \mathcal{R})$*,* $\mathcal{L}$ *contains all isomorphisms and is stable under right cancellation, composition and pushout.*

Stability under right cancellation means that if some composite $g \circ h$ *is in* $\mathcal{L}$ *for* $h \in \mathcal{L}$*, then so is* g*.*

Stability under pushout means that given any pushout square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "l"'] & B \arrow[d, "l'"] \\ C \arrow[r, "g"'] & D, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

if $l \in \mathcal{L}$*, then also* $l' \in \mathcal{L}$*.*

Dually, $\mathcal{R}$ *contains all isomorphisms and is stable under left cancellation, composition and pullback (in the obvious dual sense to stability under right cancellation and pushout).*

So in particular both classes determine identity-on-objects subcategories of $\mathcal{C}$.

Let us now consider the locally full sub-pseudo double category $\mathsf{Cospan}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$ of $\mathsf{Cospan}(\mathcal{C})$

- whose horizontal category is $\mathcal{C}$,
- and whose vertical morphisms $Y \to X$ are cospans $Y \xrightarrow{s} U \xleftarrow{l} X$ with $l \in \mathcal{L}$.

Proposition 54. $\mathsf{Cospan}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$ *forms a sub-pseudo double category of* $\mathsf{Cospan}(\mathcal{C})$ *that is fibred if* $\mathcal{C}$ *has pullbacks.*

Proof. That $\mathsf{Cospan}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$ forms a pseudo double category is a simple consequence of Lemma 53. To see that it is fibred, consider any vertical morphism $Y \xrightarrow{f} U \xleftarrow{l} X$ and horizontal morphism $X' \xrightarrow{h} X$. In order to construct a cartesian lifting of (f, l)

along h, we factor the composite $X' \xrightarrow{h} X \xrightarrow{l} U$ as $X' \xrightarrow{l'} U' \xrightarrow{h'} U$, with $l' \in \mathcal{L}$ and $h' \in \mathcal{R}$, and then take the pullback of f and h', as in the front face below:

(10)
$$
\begin{tikzcd}
Y'' \arrow[drr, bend left, "q''"] \arrow[dr, dashed, "s''"'] \arrow[dd, "f''"'] & & \\
& Y' \arrow[r, "h''"'] \arrow[dd, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[dd, "f"] \\
U'' \arrow[drr, bend left, "q'"] \arrow[dr, dashed, "s'"] & & \\
& U' \arrow[r, "h'"] & U \\
X'' \arrow[uu, "l''"] \arrow[drr, bend left, "q"] \arrow[dr, "s"'] & & \\
& X' \arrow[uu, "l'"] \arrow[r, "h"] & X \arrow[uu, "l"']
\end{tikzcd}
$$

The obtained morphism (h, h', h'') is generally not cartesian in $\mathsf{Cospan}(\mathcal{C})_H$, but let us show that it is in $\mathsf{Cospan}_{\mathcal{L},\mathcal{R}}(\mathcal{C})$. For this, consider any morphism (q, q', q'') to U such that $q = hs$ as above; then since $l'' \in \mathcal{L}$ (by hypothesis) and $h' \in \mathcal{R}$ (by construction), we obtain by the lifting property a unique s' making everything in sight commute. But then the universal property of pullback gives the desired s''. $\square$

For any signature S over some base category $\mathbb{C}$, we will try to apply this construction to the pseudo double category of plays $\mathbb{D}(\mathsf{S})$ over S, with the factorisation system generated by the set J_{S} of all "t-legs", i.e., the set of morphisms $X \xrightarrow{t} M$ for $Y \xrightarrow{s} M \xleftarrow{t} X$ spanning seeds. A map is then in $J_{\mathsf{S}}^{\perp}$ when no new move is added "forwards", i.e., following the direction of time. Indeed, recalling that each M occurring in a seed should be representable, giving a square

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "t"'] & U \arrow[d, "r"] \\
M \arrow[r, "\mu"'] & V
\end{tikzcd}
$$

amounts by Yoneda to picking a move μ in V, whose initial position X is already available in U. The map r is then in $\mathcal{R}$ when all such moves are also already in U.

Our goal now reduces to showing that $\mathbb{D}(\mathsf{S})$ is fibred as a sub-pseudo double category of $\mathsf{Cospan}_{^{\perp}(J_{\mathsf{S}}^{\perp}), J_{\mathsf{S}}^{\perp}}(\widehat{\mathbb{C}})$ – which we henceforth abbreviate to $\mathsf{Cospan}_{J_{\mathsf{S}}}(\widehat{\mathbb{C}})$. The difficulty is that, in a situation like (10), the factorisation system yields a cartesian lifting (h, h', h'') in $\mathsf{Cospan}_{J_{\mathsf{S}}}(\widehat{\mathbb{C}})$, of which we will further need to prove that (1) it lies in $\mathbb{D}(\mathsf{S})_H$, and (2) it is also cartesian there. Point (2) reduces to proving that if (q, q', q'') is in $\mathbb{D}(\mathsf{S})_H$ then so is (s, s', s'').

In fact, assuming that the candidate lifting is in $\mathbb{D}(\mathsf{S})$, its cartesianness follows from the fact that all mediating arrows, computed as in (10), are also in $\mathbb{D}(\mathsf{S})$. Indeed, we have:

Lemma 55. *$\mathbb{D}(\mathsf{S})_H$ has the left cancellation property: for all β and α in $\mathsf{Cospan}(\widehat{\mathbb{C}})$ such that $\beta \circ \alpha$ and β are in $\mathbb{D}(\mathsf{S})_H$, then also α is in $\mathbb{D}(\mathsf{S})_H$.*

Proof. By 1D-analogues of the pullback lemma and left cancellation for injectives. $\square$

It thus remains to prove that the candidate lifting is a play, and that the morphism (h, h', h'') lies in $\mathbb{D}(\mathsf{S})_H$. Let us record this as:

Lemma 56. *Assume that in all situations like* (10)*, if U is a play and h is 1D-injective, then U' is again a play and (h, h', h'') is in $\mathbb{D}(\mathsf{S})_H$. Then, $\mathbb{D}(\mathsf{S})$ is fibred.*

We thus consider conditions for this to hold. In Section 4.3, we show that if it holds for seeds, then it extends to all plays. In Section 4.4, we investigate conditions for the result to hold for seeds.

This all rests on a few elementary facts about 1D-pullbacks and 1D-injectivity in presheaf categories, which we now prove.

4.2. **A little theory of 1D-pullbacks and 1D-injectivity.** Let us start by recasting the definitions of 1D-injectivity and 1D-pullback in the following setting:

Definition 57. *A* one-way category *consists of category* $\mathbb{C}$ *equipped with a functor to* 2*, the ordinal* 2 *viewed as a category.*

Any direct category $d: \mathbb{C} \to \omega$ may be viewed as a one-way category by post-composing with the functor $\omega \to 2$ mapping everyone to 1 except 0 which is mapped to itself.

Definition 58. *Let* $d: \mathbb{C} \to 2$ *be a one-way category. The* dimension *of an object of* $\mathbb{C}$ *is its image by* d*. A natural transformation between presheaves over* $\mathbb{C}$ *is* 1D-injective *iff its components on objects of dimension* 1 *are injective. A square of natural transformations is a 1D-pullback iff it is at all objects of dimension* 1*.*

Proposition 59. *The definitions of dimension (or, more precisely, whether a dimension is equal to* 0 *or not), 1D-injectivity, and 1D-pullbacks given for direct categories* $d: \mathbb{C} \to \omega$ *coincide with their analogues for the corresponding one-way category* $\mathbb{C} \to \omega \to 2$*.*

Proof. Trivial. □

We here work in the simpler setting of presheaves over a one-way category, but by the proposition we may transport our results from the one-way categorical setting to the direct categorical one. We will do so silently in the sequel.

There are several functors from one-way categories to categories, but the important one for us restricts its argument to dimension 1:

Definition 60. *Let* $\pi_1: \mathsf{Cat}/2 \to \mathsf{Cat}$ *denote pullback along* $\ulcorner 1 \urcorner : 1 \to 2$*.*

In principle, this should rely on some global choice of pullbacks, but the easiest is to pick the pullbacks making each arrow $\mathsf{i}_1: \pi_1(\mathbb{C}) \hookrightarrow \mathbb{C}$ an inclusion.

Notation 61. *We denote* $\pi_1(\mathbb{C})$ *by* $\mathbb{C}_{|1}$*.*

We now have the standard chain of adjunctions:

$$\begin{tikzcd} \widehat{\mathbb{C}_{|1}} \arrow[rr, bend left, "\Sigma_{\mathsf{i}_1}"] \arrow[rr, bend right, "\Pi_{\mathsf{i}_1}"'] & & \widehat{\mathbb{C}}, \arrow[ll, "\Delta_{\mathsf{i}_1}" description] \end{tikzcd}$$

where Δ_{i_1}, $\sum_{\mathsf{i}_1}$ and $\prod_{\mathsf{i}_1}$ respectively denote restriction, left Kan extension and right Kan extension along the opposite of i_1.

Proposition 62. *A morphism in* $\widehat{\mathbb{C}}$ *is 1D-injective iff its image by* Δ_{i_1} *is injective. A square in* $\widehat{\mathbb{C}}$ *is a 1D-pullback iff its image by* Δ_{i_1} *is a pullback.*

Proof. By definition and the fact that limits are pointwise in presheaf categories. □

It is instructive to push things just a bit further. In particular, we establish a characterisation of 1D-injectivity and pullbacks analogous to the standard universal properties of injectivity and pullbacks, though relative to objects of $\widehat{\mathbb{C}_{|1}}$. Our first step is:

Proposition 63. *The left adjoint $\sum_{\mathsf{i}_1}$ is full and faithful, and the comonad $\sum_{\mathsf{i}_1} \circ \Delta_{\mathsf{i}_1}$ is idempotent, so that $\widehat{\mathbb{C}_{|1}}$ is a coreflective, full subcategory of $\widehat{\mathbb{C}}$.*

Proof. It is well known [22, Proposition 4.23] that the unit of the adjunction is an isomorphism when we extend and restrict along a fully-faithful functor. Furthermore, it is also well-known [25] that if the unit of an adjunction is an isomorphism, then the left adjoint is full and faithful. The comultiplication of the induced comonad is then an isomorphism by construction. □

Our characterisations will stem from the more general

Lemma 64. *Consider any full coreflection* $L\colon \mathbb{C} \underset{\perp}{\rightleftarrows} \mathbb{D} :R$. *For any small category J and functor $D\colon J \to \mathbb{D}$, if RD has a limit in $\mathbb{C}$, then $L(\lim_j RD(j))$ has the universal property of a limit of D relative to objects of $\mathbb{C}$, i.e., we have for all $X \in \mathbb{C}$:*

$$\int_{j \in J} \mathbb{D}(LX, D(j)) \cong \mathbb{D}(LX, L(\lim_j RD(j)))$$

naturally in X.

Proof. We have

$$\int_{j \in J} \mathbb{D}(LX, D(j)) \cong \int_{j \in J} \mathbb{C}(X, RD(j)) \cong \mathbb{C}(X, \lim_j RD(j)) \cong \mathbb{D}(LX, L(\lim_j RD(j))),$$

where the last step is by full faithfulness of L. □

Corollary 65. *A square in $\widehat{\mathbb{C}}$ as below left is a 1D-pullback iff for all $X \in \widehat{\mathbb{C}_{|1}}$, u and v as below right making the outer diagram commute, there is a unique mediating morphism h as shown, such that $ph = u$ and $qh = v$:*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & D \end{tikzcd} \qquad \begin{tikzcd} \sum_{\mathsf{i}_1}(X) \arrow[drr, bend left, "v"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "u"'] & & \\ & A \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & D. \end{tikzcd}$$

Proposition 66. *Consider any morphism $m\colon X \to Y$ in $\widehat{\mathbb{C}}$. The following are equivalent:*

(1) *m is 1D-injective;*
(2) *for all $f, g\colon \sum_{\mathsf{i}_1}(Z) \to X$, $mf = mg$ implies $f = g$;*
(3) *the square*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "m"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \tag{11}$$

is a 1D-pullback.

Proof. By definition, these three items are respectively equivalent to

- *(1') $\Delta_{\mathsf{i}_1}(m)$ is monic,*
- *(2') for all $f, g\colon Z \to \Delta_{\mathsf{i}_1}(X)$, $\Delta_{\mathsf{i}_1}(m)f = \Delta_{\mathsf{i}_1}(m)g$ implies $f = g$;*
- *(3') the image by Δ_{i_1} of the square (11) is a pullback.*

But now these three are well-known to be equivalent. □

4.3. **A necessary and sufficient fibredness criterion.** We now prove some basic facts about $\mathsf{Cospan}(\widehat{\mathbb{C}})$, $\mathbb{D}^0(\mathbb{C})$, and $\mathsf{Cospan}_{J_{\mathsf{S}}}(\widehat{\mathbb{C}})$, from which we derive useful results about plays, and eventually our main abstract result, namely that, under the hypothesis that seeds admit cartesian restrictions (which we investigate independently in the next section), $\mathbb{D}(\mathsf{S})$ is fibred.

Let us start with some notation:

Notation 67. *The cospan underlying any play $u: Y \twoheadrightarrow X$ will be denoted by $Y \xrightarrow{s_u} U \xleftarrow{t_u} X$ (using capitalisation for the middle object). We will often denote cospans $Y \xrightarrow{s} U \xleftarrow{t} X$ simply by $\langle U \rangle$, leaving the context provide the missing information. Furthermore, pushouts exist in $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$ and any pushout of two maps in $\mathbb{D}^0(\mathbb{C})_H$ yields a square in $\mathbb{D}^0(\mathbb{C})_H$. However, this square may not be a pushout in $\mathbb{D}^0(\mathbb{C})_H$ – because mediating arrows may not be 1D-injective. We will slightly abuse notation and mark such squares as pushouts even when considered in $\mathbb{D}^0(\mathbb{C})_H$.*

Let us start with some preliminary work about tightness.

Lemma 68. *For any position X, we have $\mathrm{Pl}(X) \cong \sum_{\mathsf{i}_1}(\Delta_{\mathsf{i}_1}(X))$ (recalling Definition 32). In particular, X is tight iff $\mathrm{Pl}(X) \to X$ is epi.*

Proof. By definition. □

Lemma 69. *Consider any diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] \arrow[ur, "l"] & D \end{tikzcd}$$

in $\widehat{\mathbb{C}}$ where only the outer square and the bottom right triangle are known to commute, i.e., $kf = hg$ and $kl = h$. If A is tight and k is 1D-injective, then also the top left triangle commutes.

Proof. Post-composing with k, we have by hypothesis that $kf = hg = klg$, hence $kf\varepsilon_A = klg\varepsilon_A$. By 1D-injectivity of k and Proposition 66, we get $f\varepsilon_A = lg\varepsilon_A$. By tightness of A and Lemma 68, we finally obtain $f = lg$. □

Definition 70. *The* cofree invariant position *of a cospan $Y \to U \leftarrow X$ is given by the pullback*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & U. \end{tikzcd}$$

The terminology is justified by the following

Proposition 71. *Fixing any global choice of pullbacks, taking the cofree invariant position Z_u of any play u induces a functor $Z_{-}: \mathsf{Cospan}(\widehat{\mathbb{C}})_H \to \widehat{\mathbb{C}_{|1}}$ which is right adjoint to the subcategory inclusion $\widehat{\mathbb{C}_{|1}} \hookrightarrow \mathsf{Cospan}(\widehat{\mathbb{C}})_H$. The inclusion being obviously full and faithful, the unit is an isomorphism, and the associated comonad is idempotent.*

Proof. By universal property of pullback. □

Lemma 72. *Any pushout in* $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$ *as below left, where Z is a position, may be factored as below right, where Z_0 is the cofree invariant position of U:*

$$\begin{tikzcd} id^\bullet_Z \arrow[r] \arrow[d] & \langle U\rangle \arrow[d] \\ \langle V\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle W\rangle \end{tikzcd} \qquad \begin{tikzcd} id^\bullet_Z \arrow[r] \arrow[d] & id^\bullet_{Z_0} \arrow[r] \arrow[d] & \langle U\rangle \arrow[d] \\ \langle V\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle V'\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle W\rangle . \end{tikzcd}$$

If $\langle V\rangle$ is isomorphic to $id^\bullet_{Z'}$ for some position Z', then $\langle V'\rangle$ is isomorphic to $id^\bullet_{Z'_0}$ for some position Z'_0.

Proof. The last point is a consequence of colimits being pointwise in presheaf categories. For the first, we get a map $Z \to Z_0$ such that $id^\bullet_Z \to \langle U\rangle = id^\bullet_Z \to id^\bullet_{Z_0} \to \langle U\rangle$ by universal property of pullback. We can then define $\langle V'\rangle$ as the pushout of $\langle V\rangle$ along $id^\bullet_Z \to id^\bullet_{Z_0}$ and obtain a unique morphism $\langle V'\rangle \to \langle W\rangle$ by its universal property: this yields a diagram as desired, whose right-hand square is again a pushout by the pushout lemma. Moreover, if $\langle V\rangle$ is isomorphic to $id^\bullet_{Z'}$ for some position Z', then we obviously have that $\langle V'\rangle$ is isomorphic to $id^\bullet_{Z'_0}$ for some position Z'_0. □

Lemma 73. *Cofree invariant positions are stable under pushout in the following sense: if Z is the cofree invariant position of $\langle U\rangle$ and*

$$\begin{tikzcd} id^\bullet_Z \arrow[r] \arrow[d] & id^\bullet_{Z'} \arrow[d] \\ \langle U\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle U'\rangle \end{tikzcd}$$

is a pushout, then Z' is the cofree invariant position of $\langle U'\rangle$.

Proof. Let us first name the involved presheaves: $\langle U\rangle = (Y \to U \leftarrow X)$ and $\langle U'\rangle = (Y' \to U' \leftarrow X')$. Since Z is the cofree invariant position of $\langle U\rangle$, we may apply Corollary 18 to

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] \arrow[dd] & & Y \arrow[dr] \arrow[dd] & \\ & Z' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] \\ X \arrow[rr, hook] \arrow[dr] & & U \arrow[dr] & \\ & X' \arrow[rr] & & U' , \end{tikzcd}$$

to obtain that the front face is also a pullback. □

Lemma 74. *In* $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$*, vertical composition preserves pushout squares. More explicitly, given two vertically composable pushouts as below left and center, the composite square below right is again a pushout:*

$$\begin{tikzcd} \langle U_0\rangle \arrow[r] \arrow[d] & \langle U_1\rangle \arrow[d] \\ \langle U_2\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle U\rangle \end{tikzcd} \qquad \begin{tikzcd} \langle V_0\rangle \arrow[r] \arrow[d] & \langle V_1\rangle \arrow[d] \\ \langle V_2\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle V\rangle \end{tikzcd} \qquad \begin{tikzcd} \langle U_0\rangle \bullet \langle V_0\rangle \arrow[r] \arrow[d] & \langle U_1\rangle \bullet \langle V_1\rangle \arrow[d] \\ \langle U_2\rangle \bullet \langle V_2\rangle \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \langle U\rangle \bullet \langle V\rangle . \end{tikzcd}$$

Proof. Let us first name the involved presheaves: $\langle U_i\rangle = (Y_i \to U_i \leftarrow X_i)$, $\langle V_i\rangle = (Z_i \to V_i \leftarrow Y_i)$, and similarly for $\langle U\rangle$ and $\langle V\rangle$. If we call Λ the posetal category with objects 0, 1, and -1, and morphisms generated by $0 < 1$ and $0 < -1$ (the "walking span" category), we introduce a bifunctor from $\Lambda \times \Lambda$ to $\widehat{\mathbb{C}}$ through the following diagram:

$$\begin{tikzcd}
V_1 & V_0 \arrow[l] \arrow[r] & V_2 \\
Y_1 \arrow[u] \arrow[d] & Y_0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y_2 \arrow[u] \arrow[d] \\
U_1 & U_0 \arrow[l] \arrow[r] & U_2.
\end{tikzcd}$$

By computing its colimit first horizontally, then vertically, we get $\langle U\rangle \bullet \langle V\rangle$, which by interchange of colimits is the desired pushout. ☐

Lemma 75. *Any morphism in $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})_H$ is cartesian iff it has the shape of the front face of* (10)*, i.e., its top square is a pullback and its middle morphism is in $J_\mathsf{S}^\perp$.*

Proof. The "if" direction follows from the proof of Proposition 54. For the "only if" direction, the considered properties are stable under composition with isomorphisms in $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})_H$. But any cartesian $\alpha\colon \langle U\rangle \to \langle U'\rangle$ is uniquely isomorphic in $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})_H/\langle U'\rangle$ to the cartesian lifting of $\langle U'\rangle$ along $\mathrm{cod}(\alpha)$ computed as in (10), hence the result. ☐

Lemma 76. *Any commuting square in $\widehat{\mathbb{C}}$ which is a pullback in dimension* 1 *satisfies the universal property of pullbacks w.r.t. tight positions. Concretely, for any commuting diagram as the solid part of*

$$\begin{tikzcd}
T \arrow[dr, dashed, "k"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right] & & \\
& A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d] \\
& B \arrow[r] & D
\end{tikzcd}$$

with the marked mono and where T is a tight position (recalling that by Notation 27 the dashed little square means pullback in dimension 1*), there exists a unique map k making the diagram commute.*

Proof. We construct in turn both dashed maps in

$$\begin{tikzcd}
\mathrm{Pl}(T) \arrow[r, dashed, "l"] \arrow[d, "\varepsilon_T"'] & A \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d] \\
T \arrow[ur, dashed, "k"] \arrow[urr, "h"'] \arrow[r] & B \arrow[r] & D:
\end{tikzcd}$$

- l follows from universal property of pullback in dimension 1;
- k then follows by tightness (which ensures that ε_T is epi) and lifting in the (Epi, Mono) factorisation system.

The construction of k however does not *a priori* ensure that $f \circ k = h$, but $f \circ k \circ \varepsilon_T = f \circ l = h \circ \varepsilon_T$ which entails the result by tightness. ☐

This yields a cute analog of the pullback lemma:

Lemma 77. *In any commuting diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

with the marked mono, 1D-pullback and pullback, the outer rectangle has the universal property of pullbacks w.r.t. tight positions.

Proof. A diagram chasing similar to the proof of the pullback lemma, using Lemma 76. □

Corollary 78. *For any seeds $Y \to M \leftarrow X$ and $S \to C \leftarrow T$, any commuting diagram as the solid part of*

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d] \arrow[dr] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
C \arrow[rr, bend right] & V \arrow[r] & M
\end{tikzcd}$$

with the marked pullback in dimension 1, such that at least one of $U \to X$ and $U \to V$ is monic, may be completed as shown.

Proof. Because M is a seed, the bottom square of $\mathsf{S}(C \to M)$ yields a morphism $T \to X$ making the diagram commute; we then conclude by the lemma. □

Lemma 79. *For any two commuting squares*

$$\begin{tikzcd}
U \arrow[r, hook, "s_U"] \arrow[d, "f_U"'] & W \arrow[d, "f"'] & V \arrow[l, hook', "s_V"'] \arrow[d, "f_V"] \\
U' \arrow[r, hook, "s_{U'}"'] & W' & V' \arrow[l, hook', "s_{V'}"]
\end{tikzcd}$$

of 1D-injective maps in $\widehat{\mathbb{C}}$, such that

- *both squares are pullbacks in dimension 1,*
- *s_U and s_V are jointly surjective and both monic,*
- *$s_{U'}$ and $s_{V'}$ are jointly surjective and both monic,*
- *f_U and f_V are in $J_{\mathsf{S}}^{\perp}$,*

then $f \in J_{\mathsf{S}}^{\perp}$.

Proof. Consider any morphism $T \to C$ in J_{S} and commuting square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & W \arrow[d] \\
C \arrow[r] & W'.
\end{tikzcd}$$

We want to show that there is a unique diagonal filler $C \to W$. Uniqueness follows from 1D-injectivity of f and the fact that $C \cong \Sigma_{\pi_1}(\Delta_{\mathrm{i}_1}(C))$, so we only need to show existence. Furthermore, by joint surjectivity and because C is a representable of dimension > 1, $C \to W'$ factors either through U' or through V' (possibly both).

Both cases being symmetric, we only treat one. If $C \to W'$ factors through U', then we get a commuting diagram as the solid part of

$$\begin{tikzcd}
T \arrow[rr] \arrow[dr, dashed, "k"'] \arrow[ddd] & & W \arrow[ddd] \\
& U \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \\
& U' \arrow[dr] & \\
C \arrow[rr] \arrow[ur] \arrow[uur, dashed, "l"] & & W',
\end{tikzcd}$$

hence a map k as indicated by Lemma 76, using monicity of s_U. By hypothesis, $U \to U'$ is in $J_{\mathsf{S}}^{\perp}$, so there is a unique map $l{:}\, C \to U$ making both triangles commute. By composing it with $U \to W$, we get the desired diagonal filler. □

Lemma 80. *Vertical composition in $\mathbb{D}^0(\mathbb{C})$ preserves $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})$-cartesianness. Explicitly, if any two vertically composable double cells of $\mathsf{Cospan}(\widehat{\mathbb{C}})$ are both in $\mathbb{D}^0(\mathbb{C})$ and $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})$, and are cartesian in the latter, then their vertical composite is again cartesian (in the latter).*

Proof. Consider any two composable vertical morphisms $\langle U\rangle = (Y \to U \leftarrow X)$ and $\langle V\rangle = (Z \to V \leftarrow Y)$, and similarly $\langle U'\rangle$ and $\langle V'\rangle$, together with cartesian double cells $\alpha\colon\langle U\rangle \to \langle U'\rangle$ and $\beta\colon\langle V\rangle \to \langle V'\rangle$. To show that the composite is cartesian, it is enough by Lemma 75 to show that it has the shape of the front face of (10), i.e., that its top square is a pullback and that $U \bullet V \to U' \bullet V'$ is right-orthogonal to J_S.

Because $\langle U\rangle \to \langle U'\rangle$ is cartesian, by Lemma 75, the left face of

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr] \arrow[dd] & & V \arrow[dr] \arrow[dd] & \\
& U \arrow[rr] \arrow[dd] & & U \bullet V \arrow[dd] \\
Y' \arrow[rr] \arrow[dr] & & V' \arrow[dr] & \\
& U' \arrow[rr] & & U' \bullet V',
\end{tikzcd}$$

is a pullback, so by two applications of Corollary 31 and Corollary 18 respectively, its front face is a 1D-pullback and its right one is a pullback. By Lemma 79, the obtained map $U \bullet V \to U' \bullet V'$ is thus in $J_\mathsf{S}^\perp$. Since $\langle V\rangle \to \langle V'\rangle$ is cartesian, by Lemma 75, the left-hand square below is a pullback, hence so is the right-hand one by the pullback lemma:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Z' \arrow[d] \\
V \arrow[r] & V'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Z \arrow[r] \arrow[d] & Z' \arrow[d] \\
U \bullet V \arrow[r] & U' \bullet V',
\end{tikzcd}$$

which concludes the proof. ☐

Lemma 81. *For any pushout*

(12)
$$\begin{tikzcd}
id^\bullet_Z \arrow[r, "id^\bullet_h"] \arrow[d, "k"'] & id^\bullet_{Z'} \arrow[d] \\
P \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & P'
\end{tikzcd}$$

in $\mathsf{Cospan}(\widehat{\mathbb{C}})_H$ where $h \in \mathbb{D}^0(\mathbb{C})(Z, Z')$ and $k \in \mathbb{D}(\mathsf{S})_H(id^\bullet_Z, P)$, P' is again a play and $P \to P'$ is cartesian and lies in $\mathbb{D}^0(\mathbb{C})_H$ (hence also in $\mathbb{D}(\mathsf{S})_H$).

Proof. The fact that $P \to P'$ lies in $\mathbb{D}^0(\mathbb{C})_H$ follows form 1D-injectivity of h and stability of monos under pushout in Set. For the rest, let us first show the desired result for moves and then extend it to arbitrary plays by induction.

Let us thus assume that $P = \langle M\rangle$ is a move. We know that any move $\langle M\rangle$ is a pushout of some seed $\langle M_0\rangle$. By Lemma 72, we may assume that it is the pushout of $\langle M_0\rangle$ along a morphism $Z_0 \to Z_0'$, where Z_0 is the cofree invariant position of $\langle M_0\rangle$. Since Z_0 is the cofree invariant position of $\langle M_0\rangle$, by Lemma 73, we know that Z_0' is the cofree invariant position of $\langle M\rangle$. Therefore, $id^\bullet_Z \to \langle M\rangle$ factors as $id^\bullet_Z \to id^\bullet_{Z_0'} \to \langle M\rangle$. Now, we define Z'' as the pushout below and $id^\bullet_{Z''} \to \langle P'\rangle$ by its universal property:

$$\begin{tikzcd}
id^{\bullet}_{Z} \arrow[r] \arrow[d] & id^{\bullet}_{Z'} \arrow[d] \arrow[ddr, bend left] & \\
id^{\bullet}_{Z'_0} \arrow[r] \arrow[dr] & id^{\bullet}_{Z''} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& \langle M \rangle \arrow[r] & \langle P' \rangle.
\end{tikzcd}$$

Now, two applications of the pushout lemma give that

$$\begin{tikzcd}
id^{\bullet}_{Z_0} \arrow[r] \arrow[d] & id^{\bullet}_{Z''} \arrow[d] \\
\langle M_0 \rangle \arrow[r] & \langle P' \rangle
\end{tikzcd}$$

is a pushout, so $\langle M' \rangle = \langle P' \rangle$ is a move.

Let us now show that the obtained morphism $\langle M \rangle \to \langle M' \rangle$ is cartesian using Lemma 75, i.e., by showing that it has the shape of the front face of (10). By the pushout lemma, the top square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Y' \arrow[d] \\
M \arrow[r] & M'
\end{tikzcd}$$

is a pushout along $Y \to M$, which is monic, so it is a pullback by adhesivity. Moreover, to show that $M \to M'$ is right-orthogonal to J_{S}, we take any $T \to C$ in J_{S} and commuting square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & M \arrow[d] \\
C \arrow[r] & M'.
\end{tikzcd}$$

Since $M \to M'$ is an isomorphism in dimensions > 1 and C is a representable of dimension > 1, $C \to M'$ can be factored uniquely as $C \to M \to M'$. Now, since T is tight and $M \to M'$ is 1D-injective, by Lemma 69, we get that the top-left triangle commutes as well, hence $C \to M$ is the desired diagonal filler.

Now that we have shown that moves are stable under pushouts of the desired form and that the resulting morphism is cartesian, we proceed to show that it is also the case for arbitrary plays by induction on $\langle P \rangle$.

If $Y \to P \leftarrow X$ contains zero moves, then the result is obvious. If $Y \to P \leftarrow X$ contains at least one move, we decompose it as $\langle M \rangle \bullet \langle U \rangle$, for some move $T \to M \leftarrow X$ and play $Y \to U \leftarrow T$ containing fewer moves than P.

Because the marked square below

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& T \arrow[r] \arrow[d] & U \arrow[d] \\
& M \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout along a monomorphism, hence a pullback, we obtain a unique dashed map as shown making everything commute. Thus, $id^{\bullet}_{Z} \to P$ factors as a vertical composite of two cells $id^{\bullet}_{Z} \to U$ and $id^{\bullet}_{Z} \to M$. By Lemma 74, the desired pushout (12) is the vertical composite of the following two pushouts:

$$
\begin{tikzcd}
id^{\bullet}_{Z} \arrow[r] \arrow[d] & id^{\bullet}_{Z'} \arrow[d] \\
\langle U \rangle \arrow[r] & \langle U' \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
id^{\bullet}_{Z} \arrow[r] \arrow[d] & id^{\bullet}_{Z'} \arrow[d] \\
\langle M \rangle \arrow[r] & \langle M' \rangle .
\end{tikzcd}
$$

Since $\langle U' \rangle$ and $\langle M' \rangle$ are plays by induction hypothesis, so is $\langle P' \rangle$. Moreover, by induction hypothesis again, $\langle U \rangle \to \langle U' \rangle$ and $\langle M \rangle \to \langle M' \rangle$ are cartesian, and therefore, so is $\langle P \rangle \to \langle P' \rangle$, by Lemma 80. □

Remark 5. *The morphism $P \to P'$ thus computed is not opcartesian in general. To see this, consider $P = \langle \Lambda_{S,q} \rangle$ for any sensible S and q, so that $Z = S$, and let $Z' = (S \rhd_i S')$ for some sensible i and S. Then P' could be denoted by $\langle \Lambda_{S,q} \rhd_i S' \rangle$. Now consider the play $P'' = \langle \beta_{S,S',i,q} \rangle$ and the map $P \to P''$ given by $\mathsf{S}(\lambda)$ (recalling (5)). There can be no map $P' \to P''$, because the final position of P' contains a player over S', an object over which P'' is empty.*

Lemma 82. *If seeds admit cartesian restrictions in $\mathbb{D}(\mathsf{S})$, then so do moves.*

Proof. Consider any move as in (8) and horizontal morphism $h: X' \to X$. We start by forming the cube

$$
\begin{tikzcd}
Z'_0 \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & Z' \arrow[dr] \arrow[dd] & \\
& X'_0 \arrow[rr] \arrow[dd, "h_0"] & & X' \arrow[dd, "h"] \\
Z_0 \arrow[rr] \arrow[dr] & & Z \arrow[dr] & \\
& X_0 \arrow[rr] & & X ,
\end{tikzcd}
\tag{13}
$$

where the dashed arrow is obtained by universal property of pullback. By the pullback lemma, the left-hand face is again a pullback. Now, by Lemma 72, we may assume that $Z_0 \to X_0$ is monic, so by adhesivity the top face of (13) is again a pushout.

By Lemma 81, the morphism $\langle \mu \rangle \to \langle M \rangle$ is cartesian. By hypothesis, we obtain a cartesian lifting of $\langle \mu \rangle$ along h_0, say $Y'_0 \to U'_0 \leftarrow X'_0$. By Lemma 81, we push the obtained lifting along $Z'_0 \to Z'$ to obtain a play $\langle U' \rangle$ and a cartesian morphism $\langle U'_0 \rangle \to \langle U' \rangle$, which induce by universal property of pushout a morphism $\langle U' \rangle \to \langle M \rangle$ in $\mathsf{Cospan}(\widehat{\mathbb{C}})$ as in

$$
\begin{tikzcd}
id^{\bullet}_{Z'_0} \arrow[rr] \arrow[dr] \arrow[dd] & & id^{\bullet}_{Z'} \arrow[dr] \arrow[dd] & \\
& \langle U'_0 \rangle \arrow[rr] \arrow[dd] & & \langle U' \rangle \arrow[dd, dashed] \\
id^{\bullet}_{Z_0} \arrow[rr] \arrow[dr] & & id^{\bullet}_{Z} \arrow[dr] & \\
& \langle \mu \rangle \arrow[rr] & & \langle M \rangle .
\end{tikzcd}
$$

We want to show that $\langle U' \rangle$ is the cartesian restriction of $\langle M \rangle$ along h. In order to do that, we first need to show that $\langle U' \rangle \to \langle M \rangle$ is a morphism of plays, i.e., that it belongs to $\mathbb{D}(\mathsf{S})_H$.

Consider now the following cubes:

$$
\begin{tikzcd}[row sep=small, column sep=small]
X'_0 \arrow[rr] \arrow[dr] \arrow[dd] & & U'_0 \arrow[dr] \arrow[dd] & \\
& X' \arrow[rr, crossing over] & & U' \arrow[dd] \\
X_0 \arrow[rr, hook] \arrow[dr] & & \mu \arrow[dr] & \\
& X \arrow[rr] \arrow[from=uu, crossing over] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
Z'_0 \arrow[rr] \arrow[dr] \arrow[dd] & & Z' \arrow[dr] \arrow[dd] & \\
& Y'_0 \arrow[rr, crossing over] & & Y' \arrow[dd] \\
Z_0 \arrow[rr] \arrow[dr] & & Z \arrow[dr] & \\
& Y_0 \arrow[rr] \arrow[from=uu, crossing over] & & Y,
\end{tikzcd}
$$

where, in the left-hand case, both pushouts are obtained by the pushout lemma. In the left-hand cube, by pointwise adhesivity and Corollary 31, we obtain that $U' \to M$ is 1D-injective and that the front and right faces are 1D-pullbacks. In the right-hand cube, Corollary 31 entails that $Y' \to Y$ is 1D-injective. This entails in particular that $\langle U' \rangle \to \langle M \rangle$ indeed is a morphism of plays.

It remains to show that $\langle U' \rangle \to \langle M \rangle$ is cartesian, for which by Lemma 75 it is sufficient to show that it has the shape of the front face of (10).

First, the upper square is a pullback by pointwise application of Lemma 17 in

$$
\begin{tikzcd}[row sep=small, column sep=small]
Y'_0 \arrow[rr] \arrow[dr] \arrow[dd] & & U'_0 \arrow[dr] \arrow[dd] & \\
& Y' \arrow[rr, crossing over] & & U' \arrow[dd] \\
Y_0 \arrow[rr] \arrow[dr] & & \mu \arrow[dr] & \\
& Y \arrow[rr] \arrow[from=uu, crossing over] & & M.
\end{tikzcd}
$$

So the only point left to show is that $U' \to M$ lies in $J_{\mathsf{S}}^{\perp}$. To show this, we consider any morphism $T \to C$ in J_{S} and commuting square

$$
\begin{tikzcd}
T \arrow[r] \arrow[d] & U' \arrow[d] \\
C \arrow[r] & M
\end{tikzcd}
$$

and show that there is a unique diagonal filler. Uniqueness follows from the fact that $U' \to M$ is 1D-injective and that $C \cong \Sigma_{\pi_1}(\Delta_{\mathrm{i}_1}(C))$, so we only need to show that there exists such a diagonal filler.

First, since $\mu \to M$ is an isomorphism in dimensions > 1 and C is a representable of dimension > 1, we know that $C \to M$ factors through $\mu \to M$ in a unique way. We now want to show that $T \to U'$ factors through $U'_0 \to U'$ in such a way that

$$
\begin{tikzcd}
T \arrow[r, dashed] \arrow[rr, bend left] \arrow[d] & U'_0 \arrow[r] \arrow[d] & U' \arrow[d] \\
C \arrow[r] \arrow[rr, bend right] & \mu \arrow[r] & M
\end{tikzcd}
$$

commutes.

Stepping back a little, let us recall a cube considered above, as below left

$$
\begin{tikzcd}
T \arrow[ddd] \arrow[ddddr] \arrow[dr, dashed] \arrow[drrr, dashed] \arrow[ddrrrr, bend left] & & & & \\
& X'_0 \arrow[rr] \arrow[dr] \arrow[ddd] \arrow[dr, phantom, very near start, "\lrcorner"] & & U'_0 \arrow[dr] \arrow[ddd] & \\
& & X' \arrow[rr, hook] \arrow[ddd] \arrow[dr, phantom, very near start, "\lrcorner"] & & U' \arrow[ddd] \\
C \arrow[drrr, bend left] & & & & \\
& X_0 \arrow[rr, hook] \arrow[dr] & & \mu \arrow[dr] & \\
& & X \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r] \arrow[d] & U'_0 \arrow[d] \\
C \arrow[r] \arrow[ur, dashed] & \mu,
\end{tikzcd}
$$

where we added the map $T \to X_0$ given by $\mathsf{S}(C \to \mu)$. By Lemma 77 on the front and left faces, using tightness of T, we obtained a unique dashed arrow $T \to X'_0$ making everything commute. In particular, we obtain a square as the solid part above right. But $U'_0 \to \mu$ is in $J_{\mathsf{S}}^{\perp}$, so there is a unique dashed diagonal map as shown making both triangles commute, which gives rise to a map $C \to U'$ by composition, hence the result. □

Finally, we state and prove our first fibredness criterion:

Theorem 83. *If seeds admit cartesian restrictions in $\mathbb{D}(\mathsf{S})$, then $\mathbb{D}(\mathsf{S})$ is fibred.*

Proof. Let us consider any play $Y \to P \leftarrow X$ and show that its cartesian restriction along $X' \to X$ in $\mathsf{Cospan}_{J_{\mathsf{S}}}(\widehat{\mathbb{C}})$ lies in $\mathbb{D}(\mathsf{S})$, which is enough by Lemma 56. We proceed by induction on $Y \to P \leftarrow X$. If it is the composite of 0 moves, then $X \to P$ and $Y \to P$ are isomorphisms and the result is obvious. If it is the composite of $n+1$ moves, then it can be decomposed as $\langle M \rangle \bullet \langle U \rangle$ for some move $T \to M \leftarrow X$ and play $Y \to U \leftarrow T$. By Lemma 82, we know that $\langle M \rangle$ admits a cartesian restriction along $X' \to X$, say $T' \to V' \leftarrow X'$. Furthermore, by induction hypothesis, $\langle U \rangle$ admits a cartesian restriction along $T' \to T$, say $Y' \to U' \leftarrow T'$. By Lemma 80, the vertical composition of $\langle V' \rangle \to \langle M \rangle$ and $\langle U' \rangle \to \langle U \rangle$ is cartesian, hence the result. □

4.4. **Cartesian lifting of seeds.** In the previous section, we have shown that $\mathbb{D}(\mathsf{S})$ is fibred as soon as seeds admit cartesian restrictions in $\mathbb{D}(\mathsf{S})$. In this section, we exhibit sufficient conditions for this to be the case. I.e., possibly under additional hypotheses, in the setting of (10), if $\langle U \rangle$ is a seed, then its restriction $\langle U' \rangle$ is a play and (h, h', h'') is a morphism of plays.

The basic idea of our proof is that there are two possible cases: either X' "contains all of" X, or it does not. In more precise terms, either h is a retraction, or it is not. In both cases, for the given seed μ, we

- first construct a candidate restriction $\langle U' \rangle$,
- prove that it is indeed a play and that the morphism $\langle U' \rangle \to \langle \mu \rangle$ is a morphism of plays,
- and then finally show that it is a cartesian lifting of $\langle \mu \rangle$ along h by showing that it has the shape of the front face of (10).

The main difference between the two cases is that, in the first one, X can be thought of as a sub-position of X', so we basically extend μ so that it is played from all of X'. By constrast, in the second case, X' does not contain X, so it is impossible to play μ from it, and the restriction consists of all the "pieces" of μ that can be played from X'.

Let us make a first hypothesis that will be useful throughout the whole proof. It is equivalent to asking that moves never erase channels, in the sense that, if a channel is in the initial position of a move, then it also is in its final position. The hypothesis is the following:

Definition 84. *A signature* S *is* persistent *when for any seed* $Y \to M \leftarrow X$*, the morphism* $Z \to X$ *from its cofree invariant position is an isomorphism in dimension* 0.

Lemma 85. *The seeds of any persistent signature admit cartesian liftings along retractions.*

Proof. Consider any such signature S. By Lemma 55, it is enough to prove that the cartesian lifting in $\mathsf{Cospan}_{J_\mathsf{S}}(\widehat{\mathbb{C}})$ lies in $\mathbb{D}(\mathsf{S})$. Now consider any seed $Y \to M \leftarrow X$ and 1D-injective retraction $h: X' \to X$. Since h is a retraction, there is a section $h': X \to X'$ such that $hh' = id_X$. We call Z the cofree invariant position of $Y \to M \leftarrow X$ and define Z' as the pullback

$$\begin{tikzcd}
Z \arrow[d, "u"'] \arrow[dr, dashed, "r'"] \arrow[rr, equal, bend left] & & \\
X \arrow[dr, "h'"'] & Z' \arrow[r, "r"] \arrow[d, "u'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "u"] \\
& X' \arrow[r, "h"'] & X
\end{tikzcd}$$

and $r': Z \to Z'$ by its universal property. As a section, it is injective. Since u is an isomorphism in dimension 0 by persistence, so is u', which entails by Lemma 12 that the left-hand square above is a pushout in dimension 0. But in dimensions > 0, h is an isomorphism, hence so is r (as the pullback of an isomorphism); thus the left-hand square is also a pushout in dimensions > 0, as the composite of a pushout with an isomorphism in the arrow category $\mathsf{Set}^{\to}$. It is thus a pushout in all dimensions, hence a pushout in $\widehat{\mathbb{C}}$.

We define the cospan $\langle M' \rangle$ as the pushout

$$\begin{tikzcd}
id^\bullet_Z \arrow[r, "id^\bullet_{r'}"] \arrow[d] & id^\bullet_{Z'} \arrow[d] \arrow[dr, "id^\bullet_r"] & \\
\langle M \rangle \arrow[r, "\alpha'"'] \arrow[drr, equal, bend right] & \langle M' \rangle \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\alpha"] & id^\bullet_Z \arrow[d] \\
& & \langle M \rangle
\end{tikzcd} \tag{14}$$

and $\alpha: \langle M' \rangle \to \langle M \rangle$ by its universal property. Now, letting $(l, k, \tilde{h}) = \alpha$ and $(l', k', \tilde{h}') = \alpha'$, we can assume without loss of generality that the initial position of $\langle M' \rangle$ is X' and that $\tilde{h}' = h'$, since both

$$\begin{tikzcd}
Z \arrow[r, "r'"] \arrow[d, "u"'] & Z' \arrow[d, "u'"] \\
X \arrow[r, "h'"'] & X' \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
id^\bullet_Z \arrow[r, "id^\bullet_{r'}"] \arrow[d] & id^\bullet_{Z'} \arrow[d] \\
\langle M \rangle \arrow[r, "\alpha'"'] & \langle M' \rangle \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

are pushouts. Similarly, by the pushout lemma, we get that

$$\begin{tikzcd}
Z' \arrow[r, "r"] \arrow[d, "u'"'] & Z \arrow[d, "u"] \\
X' \arrow[r, "h"'] & X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
id^\bullet_{I'} \arrow[r, "id^\bullet_r"] \arrow[d] & id^\bullet_I \arrow[d] \\
\langle M' \rangle \arrow[r, "\alpha"'] & \langle M \rangle \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

are pushouts, so we can assume that $\tilde{h} = h$. By Lemma 81 in the left-hand square of (14), we get that $\langle M' \rangle$ is a play. Now, by Lemma 81 in the right-hand square, we get that α is cartesian, so $\langle M' \rangle$ is the cartesian restriction of $\langle M \rangle$ along h. $\square$

When h is not a retraction, we need some more hypotheses to construct restrictions (and ensure that they are indeed cartesian).

First, we want to limit the possible interactions between moves. Though it is possible to relax some of these limitations, this would induce an explosion of the number of cases to analyse, and the proof would quickly become very intricate, which is why we decided to stick to a simple case:

Definition 86. *A signature is* monolithic *when for all morphisms of plays $f\colon (Y \to M \leftarrow X) \to (Y' \to M' \leftarrow X')$ between any two seeds, if X is not a representable, then $M = M'$.*

Remark 6. *Since the base category is direct, monolithicity ensures that, if there is a morphism f between seeds as above, and X is not a representable, then $f = (id_Y, id_M, id_X)$.*

A second hypothesis that we make says that ther should exist a "biggest part" of any move from the point of view of any player involved in it. Basically, we want to ensure that any seed $Y \to \mu \leftarrow X$ has restrictions along all morphisms of positions $h\colon X' \to X$, and we prove this property by pasting together the "biggest part" of what each player in X' sees of μ when restricted along h. We here give a notion that entails the desired property, and basically amounts to asking that seeds admit cartesian restrictions along morphisms of the form $d \to X$ with respect to other seeds (as opposed to arbitrary plays):

Definition 87. *A signature is* fragmented *iff for all seeds $Y \xrightarrow{s} \mu \xleftarrow{t} X$ and players $x\colon d \to X$, there exists a seed $Y_{M,x} \xrightarrow{s_{M,x}} M_{|x} \xleftarrow{t_{M,x}} d$ and a morphism $f_{M,x}\colon M_{|x} \to M$ in $\mathbb{C}$ such that:*

(a) *the top square of $\mathsf{S}(f_{M,x})$ is a pullback, and*
(b) *for any seed $Y'' \to M'' \leftarrow X''$ and commuting diagram as the solid part below, there is a map $M'' \to M_{|x}$ making the diagram commute:*

$$\begin{tikzcd}
M'' \arrow[dr, dashed] \arrow[drr] & & \\
X'' \arrow[u] \arrow[dr] \arrow[drr] & M_{|x} \arrow[r, "f_{M,x}"'] & M \\
& d \arrow[u] \arrow[r] & X, \arrow[u]
\end{tikzcd} \tag{15}$$

Remark 7. *Since $f_{M,x}$ is 1D-injective and M'' is representable, the morphism $M'' \to M_{|x}$ in the hypothesis above is necessarily unique.*

Lemma 88. *If S is persistent, monolithic, and fragmented, then each $\mathsf{S}(f_{M,x})$ is a cartesian lifting of $\langle M \rangle$ along x in $\mathbb{D}(\mathsf{S})$.*

Proof. By Lemma 75, it is enough to prove that the top square of $\mathsf{S}(f_{M,x})$ is a pullback and that $f_{M,x}$ in $J_{\mathsf{S}}^{\perp}$. The first point holds by *(a)*. To prove the second one, consider any morphism $T \to C$ in J_{S} and commuting square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & M_{|x} \arrow[d, "f_{M,x}"] \\
C \arrow[r] & M.
\end{tikzcd}$$

We need to show that there is a unique diagonal filler $C \to M_{|x}$. By Corollary 78, we obtain a unique morphism $T \to d$ making the diagram below left commute:

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d] \arrow[dr] & d \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] \\ C \arrow[rr, bend right] & M_{|x} \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} C \arrow[drr] & & \\ T \arrow[u] \arrow[dr] \arrow[drr] & M_{|x} \arrow[r] & M \\ & d \arrow[u] \arrow[r] & X, \arrow[u] \end{tikzcd}$$

which may be arranged as on the right to have the shape of (15). We thus conclude by *(b)*. □

Lemma 88 exhibits cartesian liftings along players $d \to X$. Let us now consider more general cases, assuming a third property saying that each player involved in a *synchronisation* M (i.e., a seed whose initial position contains several players) is related to at most one of the "biggest parts" of M, in the sense of our above explanation of fragmentedness.

Definition 89. *A signature* S *is* separated *if, for all moves* $\mu \in \mathrm{ob}(\mathbb{C}_{|\geq 2})$ *with seed* $\mathsf{S}(\mu) = (Y \to \mu \leftarrow X)$*, channels* $d \in \mathrm{ob}(\mathbb{C}_{|1})$*, players* $x_1 : d_1 \to X$ *and* $x_2 : d_2 \to X$ *and commuting squares*

$$\begin{tikzcd} d \arrow[r, "y_1"] \arrow[d, "y_2"'] & \mu_{|x_1} \arrow[d, "f_{\mu,x_1}"] \\ \mu_{|x_2} \arrow[r, "f_{\mu,x_2}"'] & \mu \end{tikzcd}$$

in $\mathbb{C}$*, we have* $x_1 = x_2$ *(and hence* $f_{\mu,x_1} = f_{\mu,x_2}$*).*

Remark 8. *Separation really only says something in the case where the diagonal* $x : d \to \mu$ *does not factor through* X*. Indeed, if it does, then by the properties of 1D-pullbacks,* x *also factors through* x_1 *and* x_2*, hence by directedness of* $\mathbb{C}$ *is in fact equal to* x_1 *and* x_2*.*

Remark 9. *Separation is related to* views *in game semantics (and indeed, in the cases we are interested in, separation is derived from what is called the axiom of views in [19]). It basically states that any player that is created in a move* M *is created by at most one player.*

When h is not a retraction, the restriction of a seed along h has a particular form that we call a "quasi-move", which basically consists of several moves played "in parallel", i.e., independently from one another.

Definition 90. *A* quasi-move *is any cospan obtained as a pushout of the form*

$$\begin{tikzcd} \sum_{i \in n} id^{\bullet}_{Z_i} \arrow[r, "id^{\bullet}_h"] \arrow[d] & id^{\bullet}_Z \arrow[d] \\ \sum_{i \in n} \langle M_i \rangle \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \langle U \rangle, \end{tikzcd}$$

in $\mathbb{D}^0(\mathbb{C})_H$*, where the* $\langle M_i \rangle$*'s are seeds.*

Lemma 91. *Every quasi-move is a play.*

Proof. Let $\langle M_i \rangle = (Y_i \xrightarrow{s_i} M_i \xleftarrow{t_i} X_i)$ for all $i \in n$. By Lemma 81, it is enough to show that $\sum_i \langle M_i \rangle$ is a play. We proceed by induction on n. If $n = 0$ the result is trivial. Otherwise, by Lemma 74, we have

$$\sum_i \langle M_i \rangle \cong (\langle M_1 \rangle + \sum_{i>1} id^{\bullet}_{X_i}) \bullet (id^{\bullet}_{Y_1} + \sum_{i>1} \langle M_i \rangle).$$

By induction hypothesis and Lemma 74 again, both components are plays, so we are done. □

We can now exhibit the construction of the restriction of a seed $Y \xrightarrow{s} M \xleftarrow{t} X$ along a morphism $h: X' \to X$ that is not a retraction.

Theorem 92. *The seeds of any persistent, monolithic, fragmented, and separated signature* S *admit cartesian liftings in* $\mathbb{D}(\mathsf{S})$.

Proof. By Lemma 85, it is enough to deal with the case where h is not a retraction. We first build the candidate restriction. By fragmentedness, we know that for each player $(d,x) \in \mathrm{pl}(X)$, there is a seed $Y_{M,x} \xrightarrow{s_{M,x}} M_{|x} \xleftarrow{t_{M,x}} d$ and a morphism $(l_{M,x}, f_{M,x}, x): \langle M_{|x} \rangle \to \langle M \rangle$. Letting $Z' = X' \times_X Z$, for any player $(d,x) \in \mathrm{pl}(X')$ we may thus construct a map $r_{M,x}$ as in

(16)
$$\begin{tikzcd}
 & & Z_{M,hx} \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddll, dashed, bend right, "r_{M,x}"'] & & Y_{M,hx} \arrow[dr] \arrow[dd] & \\
 & & & d \arrow[rr] \arrow[dd] \arrow[ddll, bend right, "x"'] & & M_{|hx} \arrow[dd] \\
Z' \arrow[rr] \arrow[dr] & & Z \arrow[rr] \arrow[dr] & & Y \arrow[dr] & \\
 & X' \arrow[rr, "h"'] & & X \arrow[rr] & & M,
\end{tikzcd}$$

where $Z_{M,hx} \to Z$ comes from the universal property of Z. We first want to show that

(17)
$$\begin{tikzcd}
\sum_{(d,x)\in \mathrm{pl}(X')} Z_{M,hx} \arrow[r, "{[r_{M,x}]_{(d,x)\in \mathrm{pl}(X')}}"] \arrow[d] & Z' \arrow[d] \\
\sum_{(d,x)\in \mathrm{pl}(X')} d \arrow[r, "{[x]_{(d,x)\in \mathrm{pl}(X')}}"'] & X'
\end{tikzcd}$$

is a pushout and that all involved maps are 1D-injective. First, it is a pullback: by Lemma 13, it suffices to show $Z_{M,hx} = d \times_{X'} Z'$ for each x, which follows by three applications of the pullback lemma in (16). Because it is a pullback of two 1D-injective maps, all of its maps are in fact 1D-injective. But then, recalling that the pullback of any isomorphism is in fact a pushout square, we have:

- in dimension 1, $\sum_{(d,x)\in \mathrm{pl}(X')} d \to X'$ is an isomorphism and
- in dimension 0, $Z' \to X'$ is an isomorphism by persistence,

so the square is a pushout in all dimensions, hence a proper pushout.

We now define our candidate restriction $\langle U' \rangle$ as the quasi-move below, and $\langle U' \rangle \to \langle M \rangle$ by its universal property:

$$\begin{tikzcd}
\sum_{(d,x)\in \mathrm{pl}(X')} id^{\bullet}_{Z'_{M,hx}} \arrow[r, "{id^{\bullet}_{[r_{M,x}]_{(d,x)\in \mathrm{pl}(X')}}}"] \arrow[d] & id^{\bullet}_{Z'} \arrow[d] \arrow[dr, "id^{\bullet}_{r}"] & \\
\sum_{(d,x)\in \mathrm{pl}(X')} \langle M_{|hx} \rangle \arrow[r, "{(l',k',h')}"'] \arrow[drr, bend right, "{[\mathsf{S}(f_{M,h(x)})]_{(d,x)\in \mathrm{pl}(X')}}"'] & \langle U' \rangle \arrow[dr, dashed, "{(l,k,\tilde{h})}"'] & id^{\bullet}_{Z} \arrow[d] \\
 & & \langle M \rangle.
\end{tikzcd}$$

First, $\langle U' \rangle$ is a quasi-move, and therefore a play by Lemma 91. Moreover, because (17) is a pushout, we can assume without loss of generality that it is the bottom square of the pushout defining $\langle U' \rangle$. Thus, $\tilde{h} = h$.

Furthermore, the morphism $[\mathsf{S}(f_{M,hx})]_{(d,x)\in \mathrm{pl}(X')}$ is in $\mathbb{D}(\mathsf{S})_H$. Indeed, that its bottom square is a 1D-pullback follows from Lemma 13; 1D-injectivity of its bottom component is 1D-injectivity of $h \circ [x]_{(d,x)\in \mathrm{pl}(X')}$; 1D-injectivity of its top component follows from that of its middle component, which itself follows from separation: the only non-trivial bit is proving it in dimension 1. So consider any players x_1 and x_2 in the domain mapped to the same player. Each of them comes from some player of X', say x'_1 and x'_2, respectively. So by Yoneda we get a commuting square

$$\begin{tikzcd}
d \arrow[r, "x_2"] \arrow[d, "x_1"'] & M_{|hx'_2} \arrow[d] \\
M_{|hx'_1} \arrow[r] & M.
\end{tikzcd}$$

By separation, we then have $hx'_1 = hx'_2$, hence $x'_1 = x'_2$ by 1D-injectivity of h. Finally, $x_1 = x_2$ follows from 1D-injectivity of f_{M,hx'_1}.

Now, $[r_{M,x}]_{(d,x)\in \mathrm{pl}(X')}$ is bijective in dimensions > 0, as the pullback of $[x]_{(d,x)\in \mathrm{pl}(X')}$, which is by construction. Thus, the map (l', k', h') is bijective in dimensions > 0, which entails that $\langle U' \rangle \to \langle M \rangle$ is again in $\mathbb{D}(\mathsf{S})_H$.

Finally, let us prove that $\langle U' \rangle$ is the restriction of $\langle M \rangle$ along h. To this end, let us first prove that k is in $J_{\mathsf{S}}^{\perp}$: consider any $T \to C$ in J_{S} and commuting square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & U' \arrow[d] \\
C \arrow[r] & M.
\end{tikzcd}$$

By Corollary 78, we get a dashed map as in

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d] \arrow[dr] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
C \arrow[rr, bend right] & U' \arrow[r] & M,
\end{tikzcd}$$

which makes the diagram commute. If C were equal to M, then $T \to X$ would be an identity, so by Lemma 36 $T \to X' \to X$ would also be the identity, which contradicts the hypothesis that h is not a retraction. Thus, C is different from M. Therefore, by monolithicity, we know that T is representable. It is in fact of dimension 1 by tightness. Hence, $T \to X'$ factors as $T = d_0 \to \sum_{(d,x)\in \mathrm{pl}(X')} d \to X'$. Now, by fragmentedness, this implies that there exists $C \to M_{|hx_0}$ making everything commute, and we find a suitable $C \to U'$ by composing it with the obvious coproduct injection $M_{|hx_0} \to \sum_{(d,x)\in \mathrm{pl}(X')} M_{|hx}$ and k'. Uniqueness of $C \to U'$ is given by 1D-injectivity of k.

Lastly, we prove that the right-hand square below is a pullback:

$$\begin{tikzcd}
Z' + \sum_{(d,x)\in \mathrm{pl}(X')} Y_{M,hx} \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
Z' + \sum_{(d,x)\in \mathrm{pl}(X')} M_{|hx} \arrow[r] & U' \arrow[r] & M.
\end{tikzcd} \tag{18}$$

By Lemma 13, the outer square is a pullback because the square below left (by Lemma 12) and each of the squares below right (by fragmentedness) are:

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, equal] & Y \arrow[d, hook] \\
Z' \arrow[r] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y_{M,x} \arrow[r] \arrow[d] & Y \arrow[d] \\
M_{|x} \arrow[r] & M.
\end{tikzcd}$$

Moreover, the left-hand square in (18) is also a pullback by Lemmas 12 and 13, and adhesivity. Since $Z' + \sum_{(d,x)\in \mathrm{pl}(X')} M_{|x} \to U'$ is epi, Lemma 14 entails that the desired square is a pullback. □

Corollary 93. *For any persistent, monolithic, fragmented, and separated signature* S*,* $\mathbb{D}(\mathsf{S})$ *is fibred.*

As an easy application, we get:

Proposition 94. $\mathbb{D}(\mathsf{S}_{HON})$ *is fibred.*

Proof. By Corollary 93, it suffices to verify that S_{HON} is persistent, monolithic, fragmented, and separated, which is routine. □

5. Application to Tsukada-Ong strategies

In this section, we apply our general results to relate the fibred pseudo double category $\mathbb{D}(\mathsf{S}_{HON})$ to TO-views and plays. As announced, this finally yields

- an equivalence of categories between both notions of behaviour,
- a functor from our notion of strategy to theirs, which restricts to an equivalence on (both subcategories of) innocent strategies.

We first define and study the relevant categories of views and plays $\mathbb{E}(X)$ and $\mathbb{E}^{\mathbb{V}}(X)$ in Section 5.1, which leads to their characterisation as subcategories of $X/\widehat{\mathbb{L}}$. We then state our main results in Section 5.2, deferring the most technical points to the next sections: in Section 5.3, we actually construct our functor F from TO-plays to plays, which is proved in Section 5.4 to be fully faithful. We then show in Section 5.5 that F restricts to a functor $F^{\mathbb{V}}$ from TO-views to views, which is an equivalence of categories.

5.1. **Views and plays.** The relevant categories of plays for us are the "relative" variants $\mathbb{E}(X)$ of our category $\mathbb{E}$ (Definition 47). We will in particular study $\mathbb{E}(A \vdash B)$ for all sequents $A \vdash B$. We now define the subcategory of *views*.

Definition 95. *For any play* $u: Z \twoheadrightarrow (A \vdash B)$*, we define the binary relation* $\prec$ *on all moves of* u *by* $m \prec_u m'$ *iff* $m \cdot s = m' \cdot t$*. When* $m \prec_u m'$*, we say that* m' causally depends *on* m*. Furthermore, we omit the subscript when clear from context.*

Let $\mathbb{E}^{\mathbb{V}}(A \vdash B)$ *denote the full subcategory of* $\mathbb{E}(A \vdash B)$ *spanning* pre-views*, i.e., those plays* $u: Z \twoheadrightarrow (A \vdash B)$

$\boxed{\textit{\textbf{Tom: A verifier! je vire "of even length"}}}$

such that the reflexive transitive closure $\prec^*$ *of* $\prec$ *is a total ordering.*

A view *is a preview of positive, even length (i.e., it is the composite of a positive, even number of moves).*

Remark 10. *The condition imposed on pre-views by asking that* $\prec^*$ *be a total order is simply that the player who plays the* n*th move in the preview was created by the* $n-1$*th move. In particular, no moves occur "in parallel" in a pre-view.*

Remark 11. *This notion of view is morally the same as in our previous sheaf models, but slightly differs from it in form, with good reason: following our usual presentation would in particular require adding a "partial" variant of each* $\Lambda_{S,q}$ *move, in which the final position* Y *would consist of only the created player. But then, letting* $S = (\Gamma \vdash A)$*, the conclusion* A *is absent from* Y*, which breaks persistency and hence our proof of fibredness. We consider this to be a significant limitation of our signature* S_{HON}*.*

Before relating our categories of views and plays with Tsukada and Ong's, we digress a little in this section to establish the announced characterisation of $\mathbb{E}(A \vdash B)$ and $\mathbb{E}^{\mathbb{V}}(A \vdash B)$, as subcategories of the slice $(A \vdash B)/\widehat{\mathbb{L}}$.

Definition 96. *Let $\mathbb{E}'(A \vdash B)$ denote the subcategory of $(A \vdash B)/\widehat{\mathbb{L}}$ spanning morphisms $t\colon (A \vdash B) \to U$ for which there exists a play $Y \xrightarrow{s} U \xleftarrow{t} (A \vdash B)$, and 1D-injective morphisms between them.*

Let $(\mathbb{E}^{\mathbb{V}})'(A \vdash B)$ denote the full subcategory of $\mathbb{E}'(A \vdash B)$ spanning views.

There is an obvious candidate functor $\mathcal{U}\colon \mathbb{E}(A \vdash B) \to \mathbb{E}'(A \vdash B)$ mapping any $Y \xrightarrow{s} U \xleftarrow{t} (A \vdash B)$ to t and any $(w, \alpha)\colon u' \to u$ to the composite $U \hookrightarrow (U \bullet W) \xrightarrow{\alpha} U'$, where (recalling Notation 67) $u = (Y \xrightarrow{s} U \xleftarrow{t} (A \vdash B))$, $u' = (Y' \xrightarrow{s'} U' \xleftarrow{t'} (A \vdash B))$, and $w = (Z \xrightarrow{s_w} W \xleftarrow{t_w} Y)$.

Lemma 97. *$\mathcal{U}$ is compatible with the equivalence relation $\sim$ (Notation 43) and yields a functor $\mathcal{U}\colon \mathbb{E}(A \vdash B) \to \mathbb{E}'(A \vdash B)$*

Proof. Assume $\gamma\colon w' \to w$ witnesses the equivalence $(w, \alpha) \sim (w', \alpha')$. By construction of $U \bullet \gamma$ we have that $U \hookrightarrow U \bullet W$ factors as $U \hookrightarrow U \bullet W' \xrightarrow{U \bullet \gamma} U \bullet W$, so

$$\begin{array}{rcl} \mathcal{U}(w', \alpha') & = & (U \hookrightarrow U \bullet W' \xrightarrow{\alpha'} U') \\ & = & (U \hookrightarrow U \bullet W' \xrightarrow{U \bullet \gamma} U \bullet W \xrightarrow{\alpha} U') \\ & = & (U \hookrightarrow U \bullet W \xrightarrow{\alpha} U') \\ & = & \mathcal{U}(w, \alpha). \end{array}$$

Functoriality of the obtained assignment is straightforward. $\square$

The rest of this section is devoted to proving:

Theorem 98. *$\mathcal{U}$ is an equivalence, and thus restricts to an equivalence $\mathcal{U}^{\mathbb{V}}\colon \mathbb{E}^{\mathbb{V}}(A \vdash B) \to (\mathbb{E}^{\mathbb{V}})'(A \vdash B)$.*

It is enough to prove that the underlying functor to $(A \vdash B)/\widehat{\mathbb{L}}$ is faithful and that its image on hom-sets precisely spans 1D-injective morphisms.

The theorem is in fact an easy consequence of:

Lemma 99. *Assume given plays $u\colon Z \multimap (A \vdash B)$ and $u'\colon Z' \multimap (A \vdash B)$, and a 1D-injective morphism $h\colon \mathcal{U}(u) \to \mathcal{U}(u')$.*

There exists a quasi-move $w\colon T \multimap Z$ and a morphism $\alpha\colon (u \bullet w) \to u'$ in $\mathbb{D}(\mathsf{S}_{HON})_H$ such that $h = (U \hookrightarrow U \bullet W \xrightarrow{\alpha} U')$, which is minimal in the sense that for any (w', α') such that $h = (U \hookrightarrow U \bullet W' \xrightarrow{\alpha'} U')$, there exists a unique $\gamma\colon w \to w'$ such that α decomposes as $\alpha' \circ (u \bullet \gamma)$, as in

$$\begin{tikzcd}
T \arrow[r] \arrow[dr, "w"'] & T' \arrow[rr, "s"] \arrow[d, "w'"] \arrow[dl, phantom, "\overset{\gamma}{\Rightarrow}"] & & Z' \arrow[dd, "u'"] \\
 & Z \arrow[d, "u"] \arrow[rr, phantom, "\overset{\alpha'}{\Longrightarrow}"] & & \\
 & (A \vdash B) \arrow[rr, equal] & & (A \vdash B).
\end{tikzcd}$$

In order to prove this, we need to analyse final positions of plays as follows. Given a position $X \in \widehat{\mathbb{L}}$, recalling Terminology 25, let us call a channel an *input* when it occurs as $x \cdot s_i$, for some player $x \in X(S)$, and an *output* when it occurs as $x \cdot t$. In general positions, channels may be both inputs and outputs, but in coproducts of sequents, each channel is one or the other, but not both.

Definition 100. *A position is* polar *when each of its channels is either an input or an output but not both.*

Non-polar positions either have disconnected channels (which are neither inputs nor outputs) or channels which are both inputs and outputs.

Definition 101. *An* interface *is a position consisting only of channels.*

Any polar position X comes with a canonical (up to isomorphism) monic map $I_X + O_X \hookrightarrow X$ from some coproduct of two interfaces I_X and O_X, surjective (hence iso) in dimension 0, such that I_X covers inputs and O_X covers outputs. Note that all maps preserve polarity (but they may also add polarities to some channels).

Lemma 102. *Any polar position admits a surjective and 1D-injective map from a coproduct of sequents, given by the counit of the comonad of Proposition 63.*

Proof. Straightforward. □

Lemma 103. *For all plays $u: Z \multimap (A \vdash B)$, Z is polar.*

Proof. This follows from the more general fact that Z is polar for any $u: Z \multimap X$ with polar X, by induction on the length of u. □

Notation 104. *We call any map as in Lemma 102 a* polar cover *of the given position. We fix a global choice of such polar covers, which, for any polar Z, we denote by* $\mathrm{Pl}(Z) = (\sum_{i \in P_Z} Z_i^P) + (\sum_{j \in N_Z} Z_j^N) \xrightarrow{\varepsilon_Z} Z$*, where each Z_i^P is a* positive *sequent, i.e., one of the form $\Gamma \vdash$, and each Z_j^N is a* negative *sequent, i.e., one of the form $\Gamma \vdash A$.*

Finally, by Lemma 81, any play u on $\mathrm{Pl}(Z)$ "descends" to a play $\varepsilon_Z \cdot u : \varepsilon_Z \cdot Y \multimap Z$, as the pushout

$$\begin{tikzcd} id^\bullet_{I_{\mathrm{Pl}(Z)}+O_{\mathrm{Pl}(Z)}} \arrow[r] \arrow[d] & id^\bullet_{I_Z+O_Z} \arrow[d] \\ u \arrow[r, "\alpha_Z^u"'] & \varepsilon_Z \cdot u. \end{tikzcd}$$

We now need a further observation on final positions. These are *a priori* associated to some play, but in fact we may pose:

Definition 105. *The* final position $\uparrow U$ *of a presheaf $U \in \widehat{\mathbb{L}}$ is the smallest subpresheaf of U containing all channels and negative players, as well as all positive players x for which there exists no move m with $m \cdot t = x$. We deem the elements of $\uparrow U$* final *in U.*

Accordingly, a morphism $h: Z \to U$ with Z a position, is called final *iff it is isomorphic to $\uparrow U$ (in $\widehat{\mathbb{L}}/U$).*

Intuitively, the final position retains only those positive players who haven't yet played any move.

Lemma 106. *For all plays $u: Y \multimap X$, s_u is final.*

Proof. By induction on u. □

Our next step will have to do with fullness of $\mathcal{U}$. It will rely on the following notion:

Definition 107. *For any play $u': Z' \multimap (A \vdash B)$, a 1D-injective morphism $k: Z \to U'$ is P-*ample *iff for all S, i, q and $m \in U'(@_{S,i,q})$, if $m \cdot t \in \mathrm{Im}(k)$ then $m \cdot s \notin \mathrm{Im}(k)$.*

Definition 108. *For any play $u': Z' \multimap (A \vdash B)$, final players of U' are called* survivors*, whilst non-final players are called* doomed*. The set of survivors of U' is denoted by* $\mathrm{Surv}(U')$ *and the set of doomed players by* $\mathrm{Doom}(U')$*.*

For any $k: Z \to U'$, we reflect the decomposition of $\mathrm{pl}(U')$ *into survivors and doomed players as* $\mathrm{pl}(Z) = \mathrm{Surv}(Z) \uplus \mathrm{Doom}(Z)$*.*

Of course, all doomed players are positive.

Notation 109. *For any play $u: Y \multimap X$, let $\lceil u \rceil$ denote the cospan $Y \xrightarrow{s_u} U = U$.*

Lemma 110. *Assume given any polar position Z, play $u': Z' \multimap (A \vdash B)$, and P-ample morphism $k: Z \to U'$.*

Then there exist quasi-seeds (= vertical morphisms which are either seeds or identities) $M_i: Y_i \multimap Z_i^P$ for all $i \in P_Z$ and a 1D-injective α_k as in

$$\begin{tikzcd}
(\sum_{i \in P_Z} Y_i + \sum_{j \in N_Z} Z_j^N) \arrow[r] \arrow[d, "M_k"'] \arrow[dr, phantom, "\xRightarrow{\alpha_Z^{M_k}}"] & \varepsilon_Z \cdot (\sum_{i \in P_Z} Y_i + \sum_{j \in N_Z} Z_j^N) \arrow[r] \arrow[d, "\varepsilon_Z \cdot M_k"] \arrow[dr, phantom, "\xRightarrow{\alpha_k}"] & Z' \arrow[d, "U'"] \\
(\sum_{i \in P_Z} Z_i^P + \sum_{j \in N_Z} Z_j^N) \arrow[r, two heads, "\varepsilon_Z"'] & Z \arrow[r, "k"'] & U',
\end{tikzcd}$$

where $M_k = (\sum_{i \in P_Z} M_i + \sum_{j \in N_Z} Z_j^N)$ and, on the right, the cospan $\lceil u' \rceil$ is viewed as a vertical morphism in $\mathsf{Cospan}(\widehat{\mathbb{L}})$.

Proof. Let $Z^N = \sum_{j \in N_Z} Z_j^N$ and $Z^P = \sum_{i \in P_Z} Z_i^P$. By Lemma 106, any negative player x in Z uniquely corresponds to some negative player x' in Z', mapped to $k(x)$ by $s_{u'}$. This yields a cell

$$\begin{tikzcd}
Z^N \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\xRightarrow{\alpha^N}"] & Z' \arrow[d, "U'"] \\
Z^N \arrow[r, hook] & U'.
\end{tikzcd}$$

Now, for any positive survivor x in Z (over some sequent S_x), if we define $Y_x = S_x$, there is a cell

$$\begin{tikzcd}
Y_x \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\xRightarrow{\alpha_x}"] & Z' \arrow[d, "U'"] \\
S_x \arrow[r, hook, "\ulcorner k(x) \urcorner"'] & U'
\end{tikzcd}$$

analogous to the above one.

Finally, for any doomed $x \in Z(S_x)$, let $x' = k(x)$ denote its image in $U'(S_x)$. Because x is doomed, there exists a (unique) move $m \in U'(@_{S_x,i,q})$ for some i and q, such that $m \cdot t = x'$. Let $(M_x: Y_x \multimap S_x) = \mathsf{S}_{HON}(@_{S_x,i,q})$ denote the seed of m. By definition of S_{HON}, Y_x is a negative sequent, and we let $y = m \cdot s \in U'(Y_x)$. By the same argument as above, y has a unique antecedent y' in Z' and so there exists a cell

$$\begin{tikzcd}
Y_x \arrow[r, "\ulcorner y' \urcorner"] \arrow[d, "M_x"'] \arrow[dr, phantom, "\xRightarrow{\alpha_x}"] & Z' \arrow[d, "U'"] \\
S_x \arrow[r, hook, "\ulcorner k(x) \urcorner"'] & U'.
\end{tikzcd}$$

By copairing all these cells, we obtain a cell $\alpha_k^0: M_k \to U'$ in $\mathsf{Cospan}(\widehat{\mathbb{L}})$, which decomposes as desired by universal property of pushout and the fact that $id^\bullet_{I_{U'}+O_{U'}}$ is the universal interface with a map to U'. It remains to prove that all involved maps are 1D-injective and that all bottom squares are 1D-pullbacks. Everything except 1D-injectivity of α_k^0 follows from Lemmas 13 and 81, together with the fact that $id^\bullet_{I_{\mathrm{Pl}(Z)}+O_{\mathrm{Pl}(Z)}} \to id^\bullet_{I_Z+O_Z}$ is bijective in dimensions > 0. Now, because k is 1D-injective, the only non-trivial possibility for a lack of 1D-injectivity of α_k^0 lies in some $x \in \mathrm{Doom}(U')$ for which the unique player $y \in Y_x$ maps in U' to some $y' = k(x')$. But in that case, the m such that $m \cdot t = k(x)$ is such that $m \cdot s = k(x') \in \mathrm{Im}(k)$, which contradicts P-ampleness of k. $\square$

From this point on until the end of the proof of Lemma 99, we will implicitly use the fact that β moves never occur in the plays we consider, which holds because we only consider plays starting from polar positions. In particular, we will use notations such as $m \cdot t$ or $m \cdot s$ for arbitrary moves m, which is ill-defined in general, but well-defined for Λ and @ moves.

We have four further handy lemmas:

Lemma 111. *For any horizontal morphism $h: X \to Y$ and play $u: Z \multimap Y$, there exists at most one cell $id^{\bullet}_X \to u$ with bottom border h.*

For any $@_{S,i,q} \in \mathrm{ob}(\mathbb{L})$, morphism $h: S \to Y$, and play $u: Z \multimap Y$, there exists at most one cell $\mathsf{S}_{HON}(@_{S,i,q}) \to u$ with bottom border h.

Proof. The first proof is straightforward. For the second, by Yoneda, any cell $\alpha: \mathsf{S}_{HON}(@_{S,i,q}) \to u$ is uniquely determined by the image of $m = id_{@_{S,i,q}}$ (which is the move element in $\mathsf{S}_{HON}(@_{S,i,q})$). But this is in fact uniquely determined by the image of $x = id_S \in \mathsf{y}_S$ (the player element). Indeed, by an easy induction on u, there is at most one $m' \in u(@_{S,i,q})$ such that $m' \cdot t = h(x)$. □

Lemma 112. *For any play $u: Z \multimap Y$ and player x in U, either x is in the image of t_u or there exists a move m in $U(@_{S,i,q})$, for some S, i, q, such that $x = m \cdot s$. Dually, either x is in the image of s_u or there exists m in U such that $x = m \cdot t$.*

Proof. By induction on u. □

Lemma 113. *For any play $u: Z \multimap Y$ and moves m_1 and m_2 in U, if $m_1 \cdot s = m_2 \cdot s$, then $m_1 = m_2$.*

Proof. By induction on u □

We now define a relation analogous to $\prec$ (Definition 95), but on players:

Definition 114. *For any players x and y in u, let $x \prec_u y$ iff there exists m such that $x = m \cdot t$ and $y = m \cdot s$. As before, we omit the subscript when clear from context.*

Lemma 115. *For all morphisms $k: \mathcal{U}(u) \to \mathcal{U}(u')$ in $\mathbb{E}'(A \vdash B)$, if $k(x) \prec_{u'} k(y)$ with x positive, i.e., over some $@_{S,i,q}$, then $x \prec_u y$.*

Proof. Let m' witness $k(x) \prec_{u'} k(y)$. By Lemma 112, we have $k(y) \notin \mathrm{Im}(t_{u'})$, so $y \notin \mathrm{Im}(t_u)$, and hence there exists $m \in U$ such that $m \cdot s = y$ again by Lemma 112. Now, by naturality of k, we have $k(m) \cdot s = k(y)$ so $k(m) = m'$ by Lemma 113. Again by naturality of k, we get $k(m \cdot t) = m' \cdot t = k(x)$, so by 1D-injectivity of k we obtain $x = m \cdot t$ and hence $x \prec y$. □

Lemma 99 now follows:

Proof of Lemma 99. Given u, u' and h, let $k = h \circ s_u$. This morphism is P-ample (Definition 107): consider any $m' \in U'(@_{S,i,q})$ with $k(x) = m' \cdot t$, and $k(y) = m' \cdot s$. Then we have $k(x) \prec_{u'} k(y)$ so by Lemma 115, there exists m witnessing $s_u(x) \prec_u s_u(y)$. But by positivity of x Lemma 106 implies $s_u(x) \not\prec_u s_u(y)$, which is a contradiction.

Since k is P-ample, we may apply Lemma 110, which yields a cell $\alpha_k = (l, r, k): w \to \lceil u' \rceil$ with $w = \varepsilon_Z \cdot M_k: T \multimap Z$ a quasi-move. This in particular ensures that the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \arrow[d] \\ U \arrow[r, "h"'] & U' \end{tikzcd}$$

commutes. By universal property of pushout, this induces a unique morphism $\alpha : u \bullet w \to u'$ such that $\alpha \circ inj^W_U = h$ and $\alpha \circ inj^U_W = \alpha_k$ (where $inj^W_U : U \hookrightarrow U \bullet W$ and $inj^U_W : W \hookrightarrow U \bullet W$ are the pushout injections). This α is 1D-injective because the images of $U \setminus Z$ and $W \setminus Z$ are disjoint in dimensions ≥ 1.

It remains to show that the pair (w, α) is minimal. Consider thus any (w', α') such that $\alpha' \circ inj^{W'}_U = h$.

By Lemma 112, a positive $x \in U(S)$ in the image of t_u is doomed iff there is a move $m \in u(@_{S,i,q})$, for some i and q, such that $x = m \cdot t$.

Now, let us observe that for any positive player x of Z, $t_w(x) \in \mathrm{Doom}(W)$ iff $k(x) \in \mathrm{Doom}(U')$. To show this, we prove that the following are equivalent:

(i) $\exists x_1 \in T, t_w(x) = s_w(x_1)$,
(ii) $\neg(\exists m \in W, t_w(x) = m \cdot t)$,
(iii) $\exists x'_1 \in Z', k(x) = s_{u'}(x'_1)$,
(iv) $\neg(\exists m' \in U', k(x) = m' \cdot t)$.

Indeed, both equivalences *(i)* $\Leftrightarrow$ *(ii)* and *(iii)* $\Leftrightarrow$ *(iv)* follow from Lemma 112, so that it is enough to show *(i)* $\Rightarrow$ *(iii)* and *(iv)* $\Rightarrow$ *(ii)*. For the former, assuming $x_1 \in T$ such that $t_w(x) = s_w(x_1)$, $x'_1 = l(x_1)$ suits our needs, as

$$s'_u(l(x_1)) = r(s_w(x_1)) = r(t_w(x)) = k(x).$$

For the latter, we show the contrapositive: assuming $m \in W$ such that $t_w(x) = m \cdot t$, $m' = r(m)$ again suits our needs, as

$$r(m) \cdot t = r(m \cdot t) = r(t_w(x)) = k(x).$$

But any doomed player $x \in Z$ maps to some doomed player in W', so

$$(x \in \mathrm{Doom}(W)) \Leftrightarrow (k(x) \in \mathrm{Doom}(U')) \Leftrightarrow (x \in \mathrm{Doom}(W')).$$

This entails that $M_k \to U'$ lifts through α', as desired, uniquely by Lemma 111. But then $w \to U'$ also lifts through α' by universal property of pushout, uniquely because $\alpha^{M_k}_Z$ is epi. $\square$

Proof of Theorem 98. Lemma 99 easily entails that the image of $\mathcal{U}$ in $(A \vdash B)/\widehat{\mathbb{L}}$ on hom-sets spans 1D-injective morphisms. Faithfulness follows from minimality of the constructed pair (w, α). $\square$

5.2. **Bridging the gap.** We thus have, for all arenas A and B, on the one hand Tsukada and Ong's inclusion $\mathsf{i}_{TO} : \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$ of TO-views into TO-plays, and on the other hand the inclusion $\mathsf{i} : \mathbb{E}^{\mathbb{V}}(A \vdash B) \hookrightarrow \mathbb{E}(A \vdash B)$ described in the previous section. We at last relate the two by constructing a functor $F : \mathbb{P}_{A,B} \to \mathbb{E}(A \vdash B)$, which restricts to a functor from TO-views to views (Lemma 150), thus yielding the announced commuting square (1), which we reproduce here for convenience:

(19)
$$\begin{tikzcd} \mathbb{V}_{A,B} \arrow[r, hook, "\mathsf{i}_{TO}"] \arrow[d, "F^{\mathbb{V}}"'] & \mathbb{P}_{A,B} \arrow[d, "F"] \\ \mathbb{E}^{\mathbb{V}}(A \vdash B) \arrow[r, hook, "\mathsf{i}"'] & \mathbb{E}(A \vdash B). \end{tikzcd}$$

The functor F is constructed in Section 5.3, and $F^{\mathbb{V}}$ at the beginning of Section 5.5.

As we show below (Theorem 144), F is full and faithful. But as the right class of an orthogonal factorisation system on categories (bijective-on-objects vs. fully faithful functors), full and faithful functors enjoy the left cancellation property. So, because both embeddings are full and faithful, $F^{\mathbb{V}}$ is again full and faithful. But we also show (Theorem 151) that it is essentially surjective on objects, hence an equivalence. By the theory of exact squares [16], which we briefly recall below, this implies the main results of this section (Corollary 116): in both approaches the innocent strategy associated to any behaviour is right Kan extension along the opposite of the inclusion of views into plays. Let us denote these functors respectively by $\prod_{\mathrm{i}}:\mathbb{E}^{\mathbb{V}}\widehat{(A \vdash B)} \to \mathbb{E}\widehat{(A \vdash B)}$ and $\prod_{\mathrm{i}_{TO}}:\widehat{\mathbb{V}_{A,B}} \to \widehat{\mathbb{P}_{A,B}}$. Similarly, let Δ_F and $\Delta_{F^{\mathbb{V}}}$ respectively denote restriction along the opposites of F and $F^{\mathbb{V}}$. Our result then reads:

Corollary 116. *For all arenas A and B, the square*

$$\begin{tikzcd}
\widehat{\mathbb{V}_{A,B}} \arrow[r, hook, "\prod_{\mathrm{i}_{TO}}"] & \widehat{\mathbb{P}_{A,B}} \\
\mathbb{E}^{\mathbb{V}}\widehat{(A \vdash B)} \arrow[u, "\Delta_{F^{\mathbb{V}}}"] \arrow[r, hook, "\prod_{\mathrm{i}}"'] & \mathbb{E}\widehat{(A \vdash B)}. \arrow[u, "\Delta_F"']
\end{tikzcd}$$

commutes up to isomorphism.

Restriction along $F^{\mathbb{V}}$ induces an equivalence between behaviours over $A \vdash B$ and TO-behaviours over (A, B).

Restriction along F induces a functor from strategies over $A \vdash B$ to TO-strategies over (A, B), which restricts to an equivalence on innocent strategies.

The moral of this result is that our views and plays faithfully represent Tsukada and Ong's. Indeed, both notions of behaviour essentially coincide and moreover, although our categories of plays are slightly richer, our innocent strategies restrict to theirs and furthermore their process of extending behaviours to innocent strategies coincides with ours up to this restriction.

As mentioned above, assuming Theorems 144 and 151, Corollary 116 directly follows from (a straightforward extension of) a well-known fact about exact squares, which we now recall.

Definition 117. *A* square *is a natural transformation as in:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & B \arrow[d, "v"] \\
C \arrow[r, "g"'] & D.
\end{tikzcd} \tag{20}$$

Any square yields by restriction a square as on the left below, and so by adjunction a further square as on the right:

$$\begin{tikzcd}
\widehat{A} \arrow[dr, phantom, "\overset{\Delta_\phi}{\Longleftarrow}"] & \widehat{B} \arrow[l, "\Delta_f"'] \\
\widehat{C} \arrow[u, "\Delta_u"] & \widehat{D} \arrow[l, "\Delta_g"] \arrow[u, "\Delta_v"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{A} \arrow[r, "\prod_f"] \arrow[dr, phantom, "\overset{\prod_\phi}{\Longleftarrow}"] & \widehat{B} \\
\widehat{C} \arrow[u, "\Delta_u"] \arrow[r, "\prod_g"'] & \widehat{D}, \arrow[u, "\Delta_v"']
\end{tikzcd}$$

where, e.g., Δ_f denotes restriction along f^{op} and $\prod_f$ denotes right Kan extension along f^{op}.

Definition 118. *A square ϕ is* exact *iff $\prod_\phi$ is an isomorphism.*

Obviously, Corollary 116 reduces to exactness of (1) (filled with the identity). In order to prove that (1) is exact, let us recall two basic results from Guitart [16]:

Lemma 119. *For any functor $f: A \to B$, the square below left is exact; furthermore, the square below right is also exact if f is fully faithful:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\overset{id_f}{\Longrightarrow}"] & B \arrow[d, equal] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\overset{id_f}{\Longrightarrow}"] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

These results appear as 1.14 (1) and (4) in Guitart's paper. The next one appears as 1.8:

Lemma 120. *Exact squares are stable under horizontal composition.*

We put these together to obtain:

Lemma 121. *Any square* (20) *in which ϕ and u are identities and v is fully faithful is exact.*

Proof. We obtain the given square as the horizontal composite

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[rr, bend left, "f"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, equal] & B \arrow[d, "v"] \\ A \arrow[r, "f"] \arrow[rr, bend right, "g"'] & B \arrow[r, "v"] & C, \end{tikzcd}$$

which is exact by the previous lemma, because both squares are exact by Lemma 119. ☐

Lemma 122. *Any square* (20) *in which ϕ is an identity, u is an equivalence, and v is fully faithful is exact.*

Proof. Similar to the previous lemma. ☐

This simply entails our main result:

Proof of Corollary 116. Theorems 144 and 151 allow us to apply the previous lemma. The rest is a direct consequence of exactness. ☐

We thus devote the rest of the paper to constructing F (Section 5.3), proving Theorem 144 (Section 5.4), and finally proving Lemma 150 and Theorem 151 (Section 5.5).

5.3. **Constructing the functor.** By left cancellation and Theorem 98, constructing F reduces to defining a fully-faithful functor $F: \mathbb{P}_{A,B} \to \mathbb{E}'(A \vdash B)$, which we will then want to restrict to views, as in:

(21)
$$\begin{tikzcd} \mathbb{V}_{A,B} \arrow[r, hook, "\mathsf{i}_{TO}"] \arrow[d, "F^{\mathbb{V}}"'] & \mathbb{P}_{A,B} \arrow[d, "F"] \\ (\mathbb{E}^{\mathbb{V}})'(A \vdash B) \arrow[r, hook, "\mathsf{i}"'] & \mathbb{E}'(A \vdash B). \end{tikzcd}$$

We first construct F on objects, which requires some preparatory notation on arenas and preplays.

Notation 123. *From this point on, we use $C_{/m}$ to denote the sub-arena of C under the move m, and only use $C \cdot m$ to denote $C_{/m}$ when m is a root of C.*

We further define $(A,B)_{/m} = A_{/m}$ when m is in A and $(A,B)_{/m} = B_{/m}$ when m is in B.

We immediately notice:

Lemma 124. *For all m, m' in $M_A + M_B$, if $m \vdash_{A \multimap B} m'$ and $\star \nvdash_A m'$, then $m' \in \sqrt{(A,B)_{/m}}$ and $(A,B)_{/m'} = (A,B)_{/m} \cdot m'$.*

Proof. Trivial. ☐

Notation 125. *In any preplay $s = (n, f, \varphi)$ and $i \in n$, let $K_s(i)$ denote the length of $\lceil s \rceil_i$, and let $\lceil s \rceil_i = (I_1^{s,i}, \ldots, I_{K_s(i)}^{s,i})$. Furthermore, let $I_0^{s,i} = 0$ for all $i \in n$.*

By recasting the definition of views in HON games in terms of $K_s(i)$ and $I_j^{s,i}$, we get a few useful relations relations. First, we can follow the $K_s(-)$'s through a view. Indeed, for all $i \in n$:

- if i is odd, then $K_s(i) = 1 + K_s(\varphi(i))$ and for all $j < K_s(i)$, $I_j^{s,i} = I_j^{s,\varphi(i)}$, with the convention that $K_s(0) = 0$;
- if i is even, then $K_s(i) = 1 + K_s(i-1)$ and for all $j < K_s(i)$, $I_j^{s,i} = I_j^{s,i-1}$.

Furthermore, the definition of views reflects into the following relations:

- $I_{K_s(i)}^{s,i} = i$;
- for all odd $k \in K_s(i)$, $I_{k-1}^{s,i} = \varphi(I_k^{s,i})$;
- for all positive, even $k \in K_s(i)$, $I_{k-1}^{s,i} = I_k^{s,i} - 1$.

We often omit both superscripts when clear from context. This in particular gives φ in terms of I, for Opponent (odd) moves. Let us consider the even case. For all $j \in K_s(i)$, j and I_j have the same parity. By P-visibility, when j is even, we have $\varphi(j) \in \lceil s \rceil_i$ and so there exists a unique $l \in K_s(i)$ such that $\varphi(j) = I_l$. By alternation, $\varphi(j)$ is odd and so l also is. Thus, there exists a unique $L_s(j) \in K_s(i)/2$ such that $l = 2L_s(j) - 1$. In summary, we have:

Lemma 126. *For all $i \in n$ and $j \in \lceil s \rceil_i$, letting $k \in K_s(i)$ be such that $I_k^{s,i} = j$, we have*

- *$\varphi(j) = I_{2L_s(j)-1}$ if j is even and*
- *$\varphi(j) = \varphi(I_k^{s,i}) = I_{k-1}^{s,i}$ if j is odd.*

Furthermore, the sequence of all I_{2l-1} for $l \in \frac{K_s(i)+1}{2}$ (where the division is understood in the integer setting and thus in particular equals $\frac{K_s(i)}{2}$ when i is even) is relevant. It consists of all odd indices in $\lceil s \rceil_i$. Let us provide some notation for this:

Notation 127. *For all maps $x : n \to X$ to some set X, let us denote by $[x(i)]_{i \in n}$ the sequence $(x(1), \ldots, x(n))$.*

So, e.g., the above subsequence of I is denoted by $[I_{2l-1}]_{l \in \frac{K_s(i)+1}{2}}$. Using this notation, we may explicitly characterise the sequents associated to each stage of s. As we will see below, for each move $i \in n$, $S^{s,i+1}$ will be the sequent of the player "created by the i'th move" in $F(s)$.

Notation 128. *By convention, we extend the definition $I^{s,i}$ with $I_0^{s,i} = 0$. Furthermore, we define $(A, B)^s_{/i} = (A, B)_{/f(i)}$, where $s = (n, f, \varphi)$. We extend this by convention to $(A, B)^s_{/0} = B$.*

Definition 129. *For any $i \in n \uplus \{0\}$, let $S^{s,i+1}$ denote the sequent defined by*

- *$A, [(A, B)^s_{/I_{2l-1}^{s,i}}]_{l \in \frac{K_s(i)+1}{2}} \vdash$ if i is odd and*
- *$A, [(A, B)^s_{/I_{2l-1}^{s,i}}]_{l \in \frac{K_s(i)}{2}} \vdash (A, B)^s_{/I_{K_s(i)}^{s,i}}$ if i is even.*

In particular, when $i = 0$, the definition yields $S^{s,1} = (A \vdash B)$.

First, let us observe:

Lemma 130. *For all $i \in |s|$, if $f(i) \notin \sqrt{A}$, then $(A, B)^s_{/i} = (A, B)^s_{/\varphi(i)} \cdot f(i)$.*

Proof. If $\varphi(i) = 0$, then $f(i) \in \sqrt{B}$, so the formula obviously holds because $(A, B)^s_{/0} = B$. Otherwise, the result is a direct application of Lemma 124, using the fact that $f(\varphi(i)) \vdash_{A,B} f(i)$. □

Which entails:

Corollary 131. *The arenas* $(A,B)^s_{/I_1},\dots,(A,B)^s_{/I_{K_s(i)}}$ *are related by*

- $(A,B)^s_{/I_k} = (A,B)^s_{/I_{k-1}} \cdot f(I_k)$ *when* k *is odd,*
- $(A,B)^s_{/I_k} = (A,B)^s_{/I_{2L_s(I_k)-1}} \cdot f(I_k)$ *when* k *is even and* $f(I_k) \notin \sqrt{A}$*, and*
- $(A,B)^s_{/I_k} = A \cdot f(I_k)$ *when* k *is even and* $f(I_k) \in \sqrt{A}$*,*

for all $k \in K_s(i)$*.*

Proof. By Lemmas 126 and 130 for the first two points. Thanks to our notation above, this even works directly when $k=1$: we know that $\varphi(I_1)=0$, so $f(I_1) \in$ *rootsB* and by definition $(A,B)^s_{/I_1} = B \cdot f(I_1)$. Accordingly, the formula yields

$$(A,B)^s_{/I_1} = (A,B)^s_{/0} \cdot f(I_1) = B \cdot f(I_1).$$

The last point is straightforward. □

We now show a few useful lemmas about HON plays and morphisms of such. The first one simply states that morphisms of HO-preplays map views to views:

Lemma 132. *If* $g\colon s \to s'$ *is a morphism of HO-preplays, then for all* $i \in |s|$*:*

- $K_{s'}(g(i)) = K_s(i)$*,*
- *for all* $j \in K_s(i)$*,* $I^{s',g(i)}_j = g(I^{s,i}_j)$*.*

Proof. Let us start with the first point, by induction on i. For the base case, we have $K_{s'}(g(0)) = K_{s'}(0) = 0 = K_s(0)$ by definition. Now, by case analysis on the parity of i and then by induction hypothesis, we have:

$$\begin{aligned} K_{s'}(g(2k+1)) &= 1 + K_{s'}(\varphi'(g(2k+1))) \\ &= 1 + K_{s'}(g(\varphi(2k+1))) \\ &= 1 + K_s(\varphi(2k+1)) \\ &= K_s(2k+1) \end{aligned} \qquad \begin{aligned} K_{s'}(g(2k+2)) &= K_{s'}(g(2k+1)+1) \\ &= 1 + K_{s'}(g(2k+1)) \\ &= 1 + K_s(2k+1) \\ &= K_s(2k+2). \end{aligned}$$

For the second point, we again proceed by induction on i. The base case trivially holds. For the induction step, consider any $i > 0$. For all $j < K_s(i)$, by taking $j' = g(\varphi(i))$ in the odd case and $l = g(i) - 1 = g(2k+2) - 1 = g(2k+1)$ in the even case, we obtain

$$I^{s',g(i)}_j = I^{s',g(l)}_j = g(I^{s,l}_j)$$

(by induction hypothesis). It thus remains to prove (using the first point):

$$I^{s',g(i)}_{K_s(i)} = I^{s',g(i)}_{K_{s'}(g(i))} = g(i) = g(I^{s,i}_{K_s(i)}),$$

as desired. □

Lemma 133. *If* $g\colon s \to s'$ *is a morphism of HO-plays, then for all* $i \in \{0\} \uplus |s|$*,* $(A,B)^{s'}_{/g(i)} = (A,B)^s_{/i}$*.*

Proof. We have $(A,B)^{s'}_{/g(i)} = (A,B)_{/f'(g(i))} = (A,B)_{/f(i)} = (A,B)^s_{/i}$. □

Lemma 134. *If* $g\colon s \to s'$ *is a morphism of HO-plays, then for all* $i \in \{0\} \uplus |s|$*,* $S^{s,i+1} = S^{s',g(i)+1}$*.*

Proof. When i is odd, then $g(i)$ is also odd, so

$$S^{s',g(i)+1} = (A, [(A,B)^{s'}_{/I^{s',g(i)}_{2l-1}}]_{l \in \frac{K_{s'}(g(i))+1}{2}} \vdash).$$

By Lemma 132, we know that $K_{s'}(g(i)) = K_s(i)$ and that for all $l \in \frac{K_{s'}(g(i))+1}{2}$, $I^{s',g(i)}_{2l-1} = g(I^{s,i}_{2l-1})$, which directly implies the result by Lemma 133. The proof is similar when i is even. □

Lemma 135. *For all preplays s, $i \in |s|$, and $j \in \lceil s \rceil_i$: $K_s(j) \leq K_s(i)$ and for all $k \in K_s(j)$, $I^{s,j}_k = I^{s,i}_k$.*

Proof. By induction on i. If i is odd and $\varphi(i) = 0$, then the result is obvious. If i is odd and $\varphi(i) > 0$, then either $j = i$, in which case the result is obvious, or $j \in \lceil s \rceil_{\varphi(i)}$, in which case $K_s(j) \leq K_s(\varphi(i)) = K_s(i) - 1$ by induction hypothesis, and for all $k \in K_s(j)$, $I^{s,j}_k = I^{s,\varphi(i)}_k = I^{s,i}_k$ by induction hypothesis and the fact that $k < K_s(i)$. If i is even, the proof follows the same pattern. □

$\boxed{\textbf{Utilisé où ?}}$

Lemma 136. *If $g : s \to s'$ is a morphism of HO-preplays, then for any positive, even $i \in |s|$, we have $L_{s'}(g(i)) = L_s(i)$.*

Proof. Since $\varphi'(g(i)) = g(\varphi(i))$, we know that

$$I^{s',g(i)}_{2L_{s'}(g(i))-1} = \varphi'(g(i)) = g(\varphi(i)) = g(I^{s,i}_{2L_s(i)-1}) = I^{s',g(i)}_{2L_s(i)-1}$$

by Lemma 132, which entails the desired result, since $k \mapsto I^{s',g(i)}_k$ is monic. □

Returning to the contruction of F on objects, in fact, we construct it on all preplays in $\mathbb{PP}_{A,B}$, by induction on their length, and eventually restrict to plays. In order for our induction step to make sense, we should make explicit a few invariants that our assignment will satisfy.

Notation 137. *In the following, we will denote $x \cdot f$ by $x \odot f$ when x is a player, and use $x \cdot f$ only when x is a move.*

For any preplay $s = (n, f, \varphi)$, the associated play $F'(s) = (\uparrow U \hookrightarrow U \xleftarrow{t} X)$ will satisfy:

(P1) The sets $U(A)$ for all arenas A are pairwise disjoint, and $\biguplus_{A \in \mathbb{A}} U(A) = n+2$; this induces a map $\mathsf{a} : n+2 \to \mathbb{A}$, where $\mathbb{A}$ denotes the set of all sub-arenas of A or B, such that for all $i \in n+2$, $i \in U(\mathsf{a}(i))$;

(P2) The sets $U(S)$ for all sequents S are pairwise disjoint, and $\biguplus_{S \in \mathbb{S}} U(S) = n+1$, where $\mathbb{S}$ denotes the set of all sequents; this induces a map $\mathsf{s} : n+1 \to \mathbb{S}$ such that for all $i \in n+1$, $i \in U(\mathsf{s}(i))$;

(P3) The sets $U(\mu)$ for all moves $\mu \in \mathrm{ob}(\mathbb{L}_{|2})$ are pairwise disjoint, and $\biguplus_\mu U(\mu) = n$; this induces a map $\mathsf{m} : n \to \mathrm{ob}(\mathbb{L}_{|2})$ such that for all $i \in n$, $i \in U(\mathsf{m}(i))$;

(P4) for all $i \in n$, $i \cdot s = i+1$ (move i creates player $i+1$);

(P5) the player $n+1$ is final in U;

(P6) for all $i \in n+1$, $\mathsf{s}(i) = S^{s,i}$.

When $n = 0$, the preplay s is mapped by definition to a cospan that is isomorphic to the identity cospan on $A \vdash B$ and that verifies the conditions above. There are infinitely many such cospans, and we (arbitrarily) choose $U \!=\!= U \Leftarrow (A \vdash B)$, where such that $U(A \vdash B) = \{1\}$, $1 \odot s_1 = 1$ and $1 \odot t = 2$. For the induction step, consider any $s = (n+1, f, \varphi)$ and assume that $\uparrow U' \hookrightarrow U' \xleftarrow{t'} X$ satisfies (P1)–(P6) for the immediate prefix s' of s.

We define $F(s) = (\uparrow U \hookrightarrow U \xleftarrow{t} (A \vdash B))$ by a specific choice of composite $U' \bullet M$, for some move $M : \uparrow U \twoheadrightarrow \uparrow U'$, itself constructed by choosing some seed $Y_0 \xrightarrow{s_0} \mu \xleftarrow{t_0} X_0$ with representable X_0 and final player $x \in U'(X_0)$, and taking the pushout

$$\begin{tikzcd} X_0 \arrow[r, "t_0"] \arrow[d, "\ulcorner x \urcorner"'] & \mu \arrow[d] \\ \uparrow U' \arrow[r] & M. \end{tikzcd}$$

The map t_0 being injective, so is the induced map $U' \to U$. We choose to make it an inclusion, which makes it entirely determined by the choice of "created" elements, i.e., of images of elements in $\mu \setminus X_0$. Because $\mu \in \mathrm{ob}(\mathbb{L}_{|2})$, we know that in all cases there is exactly one created edge, one created player, and one new move. Since the edges, players and moves of U' respectively are the elements of $n+2, n+1$ and n, we will systematically pick the created elements to be respectively $n+3, n+2$ and $n+1$. Thus, we only need to choose x and μ with x final in U' and show that (P6) is preserved. Let us proceed by case analysis:

- If $n+1$ is odd and $\varphi(n+1) = 0$, then $f(n+1) \in \sqrt{B}$ so we may pick $\mu = \Lambda_{(A \vdash B), f(n+1)}$, which leaves us with the task of picking some player in $U'(A \vdash B)$. But by (P6), $1 \in U'(S^{s,1}) = U'(A \vdash B)$, so we may simply pick 1.
- If $n+1$ is odd and $\varphi(n+1) \neq 0$, then let $i = \varphi(n+1)$. Thus, we have $f(i) \vdash_{A \multimap B} f(n+1)$, so by Lemma 124 we have $f(n+1) \in \sqrt{(A,B)^s_{/i}}$. Furthermore, by (P4), and (P6), $(i \cdot s) = (i+1)$ is in $U(S^{s',i+1})$, which, because i is even by alternation, is equally
$$U(A, [(A,B)^{s'}_{/I^{s',i}_{2l-1}}]_{l \in \frac{K_{s'}(i)}{2}} \vdash (A,B)^{s'}_{/I^{s',i}_{K_{s'}(i)}}).$$
But $I^{s',i}_{K_{s'}(i)} = i$, so $(A,B)^{s'}_{/I^{s',i}_{K_{s'}(i)}} = (A,B)^{s'}_{/i} = (A,B)^s_{/i}$, so we may pick $x = (i+1)$ and $\mu = \Lambda_{S^{s',i+1}, f(n+1)}$.
- If $n+1$ is even, then we pick $x = (n \cdot s) = n+1$ (by (P4)). Moreover, $n+1$ is in $U(S^{s',n+1})$ by (P6), which, because n is odd, is equally
$$U(A, [(A,B)^{s'}_{/I^{s',n}_{2l-1}}]_{l \in \frac{K_{s'}(n)+1}{2}} \vdash).$$
If now $f(n+1) \in \sqrt{A}$, then we pick $\mu = @_{S^{s',n+1},1,f(n+1)}$. Otherwise, by Corollary 131, taking $k = n+1$ yields
$$(A,B)^s_{/n+1} = (A,B)^s_{/I^{s,n}_{2L_s(n+1)-1}} \cdot f(n+1) = (A,B)^{s'}_{/I^{s,n}_{2L_s(n+1)-1}} \cdot f(n+1),$$
But since $n+1$ is even and $2L_s(n+1)-1 < K_s(n+1)$, $I^{s,n}_{2L_s(n+1)-1} = I^{s',n}_{2L_s(n+1)-1}$, so in particular $f(n+1) \in \sqrt{(A,B)^{s'}_{/I^{s',n}_{2L_s(n+1)-1}}}$ and, in view of the form of $S^{s',n+1}$, we may pick
$$\mu = @_{S^{s',n+1},1+L_s(n+1),f(n+1)}.$$

The chosen player is final by definition when $n+1$ is odd, and by (P5) otherwise. Finally, (P6) follows straightforwardly. For example, in the second case, since the move played is $\Lambda_{S^{s',\varphi(n+1)+1}, f(n+1)}$, we know that $n+2$ is in $U(\mu)$ for $\mu = (A, [(A,B)^{s'}_{/I^{s',\varphi(n+1)}_{2l-1}}]_{l \in \frac{K_{s'}(\varphi(n+1))}{2}}, (A,B)^{s'}_{/I^{s',\varphi(n+1)}_{K_{s'}(\varphi(n+1))}} \cdot f(n+1) \vdash) = S^{s,n+2}$ (using Corollary 131).

We now want to show that F as defined here extends to a functor. We will define the image of a morphism of HO-plays below, but we first need a few preparatory lemmas to show that this image is a natural transformation.

We first state and prove a few lemmas about the presheaf U obtained by applying the construction above.

Lemma 138. *For all preplays s, if $F'(s) = (X \to U)$, then for all $i \in |s|$:*

- *$i \in U(\Lambda_{S^{s,\varphi(i)+1}, f(i)})$ if i is odd,*
- *$i \in U(@_{S^{s,i},1,f(i)})$ if i is even and $f(i) \in \sqrt{A}$,*

- *$i \in U(@_{S^{s,i},1+L_s(i),f(i)})$ if i is even and $f(i) \notin \sqrt{A}$.*

Proof. By induction on s:

- if s is empty, then the result is obvious,
- otherwise, there are two possible cases. Either $i < |s|$, in which case, if we denote by s' the immediate prefix of s, and $F'(s') = (X \to U')$, we know by induction hypothesis that i belongs to $U'(\mu) \subseteq U(\mu)$ for the desired μ (because $S^{s,i} = S^{s',i}$ and $L_s(i) = L_{s'}(i)$). Or $i = |s|$, in which case the result holds by construction.

□

Lemma 139. *For all preplays s, if $F'(s) = (X \to U)$, then for all $i \in |s|$, $i + 2 \in U((A,B)^s_{/I^{s,i}_{K_s(i)}})$.*

Proof. Like in the proof of Lemma 138, we proceed by induction on s: the result is obvious when s is empty, follows from $I^{s',i}_{K_{s'}(i)} = I^{s,i}_{K_s(i)}$ for $i < |s|$, and holds by Definition 129 and Corollary 131 for $i = |s|$. □

We can now state and prove the lemma we are interested in:

Lemma 140. *If $g: s \to s'$ is a morphism of HO-plays, $F'(s) = (X \to U)$, and $F'(s') = (X \to U')$, then:*

- *for all $\mu \in \mathbb{L}_{|2}$, if $i \in U(\mu)$, then $g(i) \in U'(\mu)$,*
- *for all $S \in \mathbb{L}_{|1}$, if $i + 1 \in U(S)$, then $g(i) + 1 \in U'(S)$,*
- *for all $C \in \mathbb{L}_{|0}$, if $i + 2 \in U(C)$, then $g(i) + 2 \in U'(C)$.*

Proof. For the first point, let us assume that i is odd. We know that if i is in $U(\mu)$, then by Lemma 138 we have $\mu = \Lambda_{S^{s,\varphi(i)+1},f(i)}$. But then $S^{s,\varphi(i)+1} = S^{s',g(\varphi(i))+1} = S^{s',\varphi'(g(i))+1}$ by Lemma 134 and the fact that $g\varphi = \varphi' g$, so $\mu = \Lambda_{S^{s',\varphi'(g(i))+1},f'(g(i))}$ (since $f'(g(i)) = f(i)$), so by Lemma 138 we have $g(i) \in U'(\mu)$. The proof is similar in the other cases (note that, in particular, the proof uses Lemma 136 when i is even and $f(i) \notin \sqrt{A}$).

The proofs of the other two points follow a similar pattern. □

We will use the following notation to define $F'(g)$:

Definition 141. *If f is a function from n to m, we define $\tilde{f}: n + 1 \to m + 1$ by $\tilde{f}(1) = 1$ and $\tilde{f}(i + 1) = f(i) + 1$ for all $i \in n$.*

Now, if $g: s \to s'$ is a morphism of HO-plays, $F'(s) = (X \to U)$, and $F'(s') = (X \to U')$, we define $k = F'(g): U \to U'$ as:

- $k_\mu(x) = g(x)$ for all $\mu \in \mathbb{L}_{|2}$ and $x \in U(\mu)$,
- $k_S(x) = \tilde{g}(x)$ for all $S \in \mathbb{L}_{|1}$ and $x \in U(S)$,
- $k_A(x) = \tilde{\tilde{g}}(x)$ for all $A \in \mathbb{L}_{|0}$ and $x \in U(A)$.

This is indeed well-defined by Lemma 140. We need to prove two other lemmas in order to show that k is natural.

Lemma 142. *For all preplays s, $i \in |s|$, if $F'(s) = (X \to U)$, then:*

- *$i \cdot t = i$ if i is even,*
- *$i \cdot t = \varphi(i) + 1$ if i is odd.*

Proof. By induction on s and i, where the only intersting case is when $i = |s|$, in which case both points hold by construction. □

By (P4), the player created by each move i is $i+1$. The next lemma describes how its associated sequent, $S^{s,i+1}$ (Definition 129), connects to its environment. Intuitively, this is quite simple: all non-new connections are as in $S^{s,i}$; and the new formula connects to the created edge, $i+2$. The technical statement is obfuscated by index handling, so let us explain this a bit. By construction, if i is odd, $S^{s,i+1}$ is positive and has $1+\frac{K_s(i)+1}{2}$ hypotheses. If i is even, $S^{s,i+1}$ is negative and has $1+\frac{K_s(i)}{2}$ hypotheses. In both cases, the first hypothesis is A, and the others have been created by previous moves. In fact, when i is even, so is $K_s(i)$, hence as the division is Euclidean, $1+\frac{K_s(i)}{2} = 1+\frac{K_s(i)+1}{2}$. So for all $i \in n$, $S^{s,i+1}$ has $1+\frac{K_s(i)+1}{2}$ hypotheses.

Lemma 143. *For all preplays s, $i \in |s|$, if $F'(s) = (X \to U)$, then, recalling Definition 129 and the fact that $i \cdot s = i+1$, we have:*

- *if i is odd:*
  - $(i+1) \odot s_j = (\varphi(i)+1) \odot s_j$ *for all* $j \in \frac{K_s(i)+1}{2}$ *and*
  - $(i+1) \odot s_{\frac{K_s(i)+1}{2}+1} = i+2;$
- *if i is even:*
  - $(i+1) \odot s_j = i \odot s_j$ *for all* $j \in \frac{K_s(i)}{2}+1,$
  - $(i+1) \odot t = i+2.$

Proof. By induction on s and i, where the only intersting case is when $i = |s|$, in which case all points hold by construction. □

We can now prove that k defined above is a natural transformation, which amounts to showing that all diagrams of the form

$$\begin{array}{ccc} U(\mu) & \xrightarrow{k_\mu} & U'(\mu) \\ {\scriptstyle U(\alpha)}\downarrow & & \downarrow{\scriptstyle U'(\alpha)} \\ U(S) & \xrightarrow[k_S]{} & U'(S) \end{array} \text{ or } \begin{array}{ccc} U(S) & \xrightarrow{k_S} & U'(S) \\ {\scriptstyle U(\beta)}\downarrow & & \downarrow{\scriptstyle U'(\beta)} \\ U(C) & \xrightarrow[k_C]{} & U'(C) \end{array} \tag{22}$$

commute, for all $\alpha: S \to \mu$ and $\beta: C \to S$. The left-hand side diagram can now be shown to commute using Lemma 142, and the right-hand side one using Lemma 143 within an induction. For example, the following computation show that the right-hand side diagram commutes for $i > 1$ odd and $\beta = s_j$ ($i = 1$ can easily be verified by hand):

$$\begin{aligned}
\tilde{\tilde{g}}(i \odot s_j) &= \tilde{\tilde{g}}((\varphi(i-1)+1) \odot s_j) && \text{by Lemma 143} \\
&= \tilde{g}(\varphi(i-1)+1) \odot s_j && \text{by induction hypothesis} \\
&= (g(\varphi(i-1))+1) \odot s_j && \\
&= (\varphi'(g(i-1))+1) \odot s_j && \text{because } g\varphi = \varphi' g \\
&= (g(i-1)+1) \odot s_j && \text{by Lemma 143} \\
&= \tilde{g}(i) \odot s_j .
\end{aligned}$$

The other points either follow a similar pattern or are proved directly by Lemma 142 or 143.

When restricted to X, k turns into id_X, as desired. Finally, k is 1D-injective by injectivity of g, which ends the definition of F on morphisms.

It remains to prove functoriality of F, which follows directly from functoriality of $\tilde{\tilde{-}}$.

5.4. **Full faithfulness.** This section is devoted to proving

Theorem 144. *The functor $F': \mathbb{P}_{A,B} \to \mathbb{E}'(A \vdash B)$ is a full embedding.*

First, F is clearly injective on objects, and faithfulness is easy: if g and g' are two morphisms $s \to s'$, then, letting $k = F'(g)$ and $k' = F'(g')$, we know by (P3) that the $U(\mu)$'s form a partition of n, and by definition of k and k' that $k_\mu(x) = g(x)$ and $k'_\mu(x) = g'(x)$ for all $x \in F'(s)(\mu)$ and $\mu \in \mathbb{L}_{|2}$. Therefore, if $k_\mu = k'_\mu$ for all $\mu \in \mathbb{L}_{|2}$, then $g = g'$.

Proving fullness is a bit more involved. Let us start with a lemma asserting that any natural transformation $k: F'(s) \to F'(s')$ is layered just as any $F'(g)$. In order to formalise this, let us state:

Definition 145. *For any TO-play s and $i \in \{0,1,2\}$, let $F'(s)_{|i} = \biguplus_{O \in \mathbb{L}_{|i}} F'(s)(O)$.*
For any $k: F'(s) \to F'(s')$ in $\mathbb{E}'(A \vdash B)$ and $i \in \{0,1,2\}$, let

$$k_{|i}: F'(s)_{|i} \to F'(s')_{|i}$$

denote the restrictions of k to each given domain and codomain.

When $U = F'(s)$ for some $s = (n, f, \varphi)$, we know by construction that $F'(s)_{|2} = n$, $F'(s)_{|1} = n+1$, and $F'(s)_{|0} = n+2$. This allows us to state:

Lemma 146. *For any $k: F'(s) \to F'(s')$ in $\mathbb{E}'(A \vdash B)$,*

$$k_{|1} = \widetilde{k_{|2}} \qquad \text{and} \qquad k_{|0} = \widetilde{\widetilde{k_{|2}}}.$$

Proof. Because $\mathbb{E}'(A \vdash B)$ is a subcategory of the coslice $(A \vdash B)/\widehat{\mathbb{L}}$, we know that $k_{|1}(1) = 1$, $k_{|0}(1) = 1$, and $k_{|0}(2) = 2$. It thus remains to prove that $k_{|1}(i+1) = k_{|2}(i) + 1$ and $k_{|0}(i+2) = k_{|1}(i+1) + 1$, for all $i \in n$.

We have

$$\begin{aligned} k_{|1}(i+1) &= k_{|1}(i \cdot s) \\ &= k_{|2}(i) \cdot s \qquad \text{(by naturality of } k\text{)} \\ &= k_{|2}(i) + 1 \end{aligned}$$

for the first point. We treat the second point by case analysis on the parity of i, using naturality and Lemma 143:

$$\begin{aligned} k_{|0}(2i+1) &= k_{|0}((2i) \odot s_N) \\ &= k_{|1}(2i) \odot s_N \\ &= k_{|1}(2i) + 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} k_{|0}(2i+2) &= k_{|0}((2i+1) \odot t) \\ &= k_{|1}(2i+1) \odot t \\ &= k_{|1}(2i+1) + 1 \end{aligned}$$

where in the odd case $N = \frac{K_s(i)+1}{2} + 1$. ☐

Consider any $s = (n, f, \varphi)$ and $s' = (n', f', \varphi')$, with $F(s) = \uparrow U \to U \leftarrow X$ and $F(s') = \uparrow U' \to U' \leftarrow X$, together with a morphism $k: U \to U'$ in $\mathbb{E}'(A \vdash B)$. We know by (P3) that the $U(\mu)$'s form a partition of n for μ in $\mathbb{L}_{|2}$. We can therefore define the map $g: n \to n'$ by $g = k_{|2}$. Our goal is to show that g is a morphism $s \to s'$ of TO-plays and that $F'(g) = k$. But given the first point, the latter follows from $g = k_{|2}$ by definition of F' and Lemma 146.

So let us show that g is a morphism of TO-plays. First, $f'(g(i)) = f(i)$ holds by construction. Furthermore, g is injective by 1D-injectivity of k. We also know that k is natural, so we get commuting diagrams as in (22) for all $\alpha: S \to \mu$ and $\beta: C \to S$. By taking $\alpha = s$, we get that $g(2i-1) = g(2i) - 1$ and $g(\varphi(2i-1)) = \varphi'(g(2i-1))$ for all $i \in n/2$. The last point we need to show is that $g(\varphi(2i)) = \varphi'(g(2i))$ for all $i \in n/2$, which requires three additional lemmas.

Lemma 147. *For all preplays s, $i \in |s|$, and $j \in \lceil s \rceil_i$, and $k \in \frac{K_s(j)+1}{2} + 1$: $(j+1) \odot s_k = (i+1) \odot s_k$.*

Proof. By induction on $i - j$ and Lemma 143. ☐

The next lemma expresses the justifier $\varphi(i)$ of i in terms of $F'(s)$, using the fact that moves and created edges are in bijection, each move i creating edge $i+2$. The idea here is that if some positive move i is played on a sequent $A_1, \ldots, A_N \vdash$, say on A_k, then $\varphi(i)$ should be the move that has created A_k. But, i being positive, the involved player is just i itself, and the corresponding edge is $i \odot s_k$, which has been created by move $i \odot s_k - 2$. There is one exception though: when $k = 1$, the move is played on A, so by definition of $A \multimap B$ its justifier is the first move played on B in the corresponding view. But this is precisely the move having created $i \odot s_2$, i.e., $i \odot s_2 - 2$.

Lemma 148. *For all preplays s, if $F'(s) = (X \to U)$ and $i \in U(@_{S,k,m})$, then:*

- $\varphi(i) = i \odot s_k - 2$ *if* $k > 1$,
- $\varphi(i) = i \odot s_2 - 2$ *if* $k = 1$.

Proof. Let us assume $k > 1$ (the other case is similar). By construction, $k = 1+L_s(i)$, but on the one hand $\varphi(i) = I^{s,i}_{2L_s(i)-1}$, and on the other hand $\varphi(i) = I^{s,\varphi(i)}_{K_s(\varphi(i))} = I^{s,i}_{K_s(\varphi(i))}$, so $2L_s(i) - 1 = K_s(\varphi(i))$, since $k \mapsto I^{s,i}_k$ is monic. Therefore, we have $k = \frac{K_s(\varphi(i))+1}{2} + 1$, so since $\varphi(i) \in \lceil s \rceil_{i-1}$, Lemma 147 entails that $(\varphi(i) + 1) \odot s_k = (i+1) \odot s_k$. By Lemma 143, since i is even, we also have $(i+1) \odot s_k = i \odot s_k$. Thus:

$$(\varphi(i) + 1) \odot s_k = i \odot s_k.$$

On the other hand, by Lemma 143 again, since $k = \frac{K_s(\varphi(i))+1}{2} + 1$,

$$(\varphi(i) + 1) \odot s_k = \varphi(i) + 2.$$

We thus derive $\varphi(i) + 2 = i \odot s_k$, hence the result. ☐

Returning to the proof that $g(\varphi(2i)) = \varphi'(g(2i))$, let us consider the case where $\mathsf{m}(2i) = @_{S,k,m}$ and $k > 1$ (the proof is similar for $k = 1$). Because $k > 1$, $\varphi(2i)$ is odd, so $\varphi(2i) \geq 1$ and hence $(2i) \odot s_k = \varphi(2i) + 2 \geq 3$. Thus, by definition, $\tilde{\tilde{g}}((2i) \odot s_k) = g((2i) \odot s_k - 2) + 2$. This entails:

$$\begin{aligned}
g(\varphi(2i)) &= g((2i) \odot s_k - 2) && \text{(by Lemma 148)} \\
&= \tilde{\tilde{g}}((2i) \odot s_k) - 2 && \text{(as we just saw)} \\
&= \tilde{\tilde{k}}_{|2}((2i) \odot s_k) - 2 && \text{(because } g = k_{|2} \text{ by definition)} \\
&= k_{|0}((2i) \odot s_k) - 2 && \text{(by Lemma 146)} \\
&= k_{|1}(2i) \odot s_k - 2 && \text{(by naturality)} \\
&= \tilde{g}(2i) \odot s_k - 2 \\
&= \tilde{g}(2i - 1 + 1) \odot s_k - 2 \\
&= (g(2i - 1) + 1) \odot s_k - 2 && \text{(by definition of } \tilde{-}) \\
&= (g(2i) - 1 + 1) \odot s_k - 2 && \text{(by } g(2i) = g(2i-1) + 1) \\
&= g(2i) \odot s_k - 2 \\
&= \varphi'(g(2i)) && \text{by Lemma 148 and } g(2i) \in F'(s')(@_{S,k,m}).
\end{aligned}$$

This proves that g is indeed a morphism of HO-plays, hence ends the proof of Theorem 144.

5.5. **Restriction to views.** In this final section, we show how our functor $F\colon \mathbb{P}_{A,B} \to \mathbb{E}'(A \vdash B)$ restricts to an equivalence on views, which achieves the construction of our candidate exact square (21).

We start with:

Lemma 149. *If $Y \to U \leftarrow (A \vdash B)$ is a play, then $\prec^*_u$ (Definition 95) is an order.*

Proof. It suffices to show that $<^*$ is antisymmetric, which we do by induction on U. If $U \cong id_{A\vdash B}$, then the result is direct. Otherwise, it is the composite of a move $Y \to M \leftarrow Y'$ and a play $Y' \to U' \leftarrow (A \vdash B)$, and $<^*_{U'}$ is an order by induction hypothesis. Now, it suffices to notice that $<^*_U$ is an extension of $<^*_{U'}$, and that adding the move M can only add pairs $x < y$ in which y is an element that is not in U', but there is only one such element, so this cannot break antisymmetry of $<^*_U$. □

This allows us to show:

Lemma 150. *The functor F restricts to a functor $F^{\mathbb{V}}\colon \mathbb{V}_{A,B} \to (\mathbb{E}^{\mathbb{V}})'(A \vdash B)$.*

Proof. Let $s = (n, f, \varphi)$ be any view. By Lemma 149, $<^*_v$ is an order. Therefore, to show that it is a total order, it is enough to show that it is a total preorder. But for all $i \in n-1$, using Lemma 142:

- if i is odd: $(i+1) \cdot t = i+1 = i \cdot s$,
- if i is even: $(i+1) \cdot t = \varphi(i+1)+1$, but because s is a view and $i+1$ is odd, $\varphi(i+1) = i$, hence $(i+1) \cdot t = i+1 = i \cdot s$.

In both cases, we have $i < i+1$ as desired. □

As alluded to before, the properties of fully-faithful functors entail that $F^{\mathbb{V}}$ is fully faithful. If we prove that it is also essentially surjective, we will obtain:

Theorem 151. *The restriction $F^{\mathbb{V}}\colon V_{A,B} \to (\mathbb{E}^{\mathbb{V}})'(A \vdash B)$ is an equivalence.*

Let us start by showing that any preview $V \leftarrow (A \vdash B)$ in $(\mathbb{E}^{\mathbb{V}})'$ is isomorphic to some *canonical* preview, in the following sense.

Definition 152. *A preview V of length n is* canonical *iff*

(i) *it satisfies points (P1)–(P3);*
(ii) *the element of $V(A \vdash B)$ corresponding to $(A \vdash B) \to V$ via Yoneda is 1, with $1 \odot s_1 = 1$ and $1 \odot t = 2$;*
(iii) *for all $i \in n$, $i \cdot t = i$, $i \cdot s = i+1$, and $i \cdot \nu = i+2$,*

where $i \cdot \nu$ denotes the channel created by move i, i.e.,

- *if $i \in V(@_{S,k,m})$, then $i \cdot \nu = i \cdot s \odot t$, whereas*
- *if $i \in V(\Lambda_{S,m})$, then $i \cdot \nu = i \cdot s \odot s_{|S|+1}$ (with $|A_1, \ldots, A_m \vdash C| = m$).*

In words, a preview is canonical when the ith move is represented by i and played by i, and its created player and channel are respectively $i+1$ and $i+2$.

Of course, we have:

Lemma 153. *For all views s, $F'(s)$ is canonical.*

Proof. By induction on s. □

Lemma 154. *Any preview is isomorphic to a unique canonical preview.*

Proof. By induction on the given preview. □

Lemma 155. *In any canonical preview V, for any sequent S and player $i \in V(S)$, we have $|S| = i/2 + 1$ and furthermore for all $0 \leq k < |S|$, $i \odot s_{k+1} = 2k+1$.*

Proof. By induction on V. □

Proof of Theorem 151. The lemma allows us to restrict attention to the claim that all canonical previews have antecedents in TO-previews (i.e., TO-preplays $s = (n, f, \varphi)$ such that $\lceil s \rceil_n = s$). We again proceed by induction on the given canonical view V. If V is the identity preview, then an antecedent is given by the empty TO-preview. Now, assume V' is any canonical preview of length n and $V = V' \bullet M$ is a canonical preview. By induction hypothesis, we get $s' = (n, f', \varphi')$ such that $F'(s') = V'$.

We will define our candidate antecedent s for V by case analysis on M. In both cases, we will first need to show that s is indeed a TO-preview, and then that $F'(s) = V$. By construction, $F'(s)$ will be determined by picking a player in V' (i.e., a valid index in $n+1$) and a valid move object in $\mathbb{L}$. By canonicity of V, it will thus be enough to show that these correspond to those of V.

There are two cases, depending on M:

- if $(n+1) \in V(\Lambda_{S,m})$ with $S = (A_1, \dots, A_N \vdash C)$ and $m \in \sqrt{C}$, then we define $s = (n+1, f, \varphi)$ as the extension of s' by
  - $f(n+1) = m$,
  - $\varphi(n+1) = n$.

  Let us first show that s is a TO-preview. Alternation is trivial, so we only need to verify that s is a justified sequence equal to its view:
  - $\varphi(n+1) < n+1$ holds trivially;
  - if $\varphi(n+1) = 0$, then $f(n+1) \in \sqrt{B}$: if $\varphi(n+1) = 0$, then by definition $n = 0$, so $C = B$ and $f(n+1) = m \in \sqrt{B}$, as desired;
  - if $\varphi(n+1) \neq 0$, then $f(\varphi(n+1))$ is the parent of $f(n+1)$ in $A \multimap B$: indeed, in that case, $n \in V(@_{(A_1,\dots,A_N \vdash),k,m'})$, for some $k \in N$ and $m' \in \sqrt{A_k}$, and $C = A_k \cdot m'$;
  - finally, $\lceil s \rceil_{n+1} = \{n+1\}$ if $\varphi(n+1) = 0$, and otherwise $\lceil s \rceil_{n+1} = \lceil s' \rceil_n \cup \{n+1\} = s' \cup \{n+1\}$, as desired.

  By construction, the image of s under F' is obtained by adding a new move, $n+1$, to $F'(s')$ (i.e., to V' by induction hypothesis), with $(n+1) \cdot t = n+1$, of the form $\Lambda_{S_{s',n+1},m}$. This agrees with V, so $F'(s) = V$, as desired.
- if $(n+1) \in V(@_{S,k,m})$, with $S = (A_1, \dots, A_N \vdash)$ and $m \in \sqrt{A_k}$, then we define $s = (n+1, f, \varphi)$ as the extension of s' by
  - $f(n+1) = m$,
  - $\varphi(n+1) = (n+1) \odot s_k - 2$ if $k > 1$,
  - $\varphi(n+1) = (n+1) \odot s_2 - 2$ if $k = 1$.

  Let us first show that s is a TO-preview. Again alternation is trivial, so we only need to verify that s is a justified sequence equal to its view:
  - $\varphi(n+1) < n+1$: if $k = 1$, then by Lemma 155 we have $\varphi(n+1) = (n+1) \odot s_2 - 2 = 2 + 1 - 2 = 1 < n+1$ because $n+1$ is even; otherwise, by Lemma 155 again we have
  $$\varphi(n+1) = (n+1) \odot s_k - 2 = 2(k-1) + 1 - 2 = 2k - 3$$
  and $N = (n+1)/2 + 1$; but $k \leq N$, so $\varphi(n+1) \leq 2((n+1)/2 + 1) - 3 = n < n+1$;
  - if $\varphi(n+1) = 0$, then $f(n+1) \in \sqrt{B}$: we in fact have $\varphi(n+1) \neq 0$, because as we saw just before
    * if $k = 1$, then by Lemma 155 $\varphi(n+1) = 1$, and
    * if $k > 1$, then by Lemma 155 $\varphi(n+1) = 2k - 3 > 1$;
  - if $\varphi(n+1) \neq 0$, then $f(\varphi(n+1))$ is the parent of $f(n+1)$ in $A \multimap B$: indeed, in that case, letting $p = \varphi(n+1)$, we have $p \in V(\Lambda_{(A_1,\dots,A_{k-1} \vdash C),m'})$, for some $m' \in \sqrt{C}$, with $A_k = C \cdot m'$; but then $f(\varphi(n+1)) = f(p) = m'$ is indeed the parent of $f(n+1) = m \in \sqrt{A_k}$;
  - finally, $\lceil s \rceil_{n+1} = \lceil s' \rceil_n \cup \{n+1\} = s' \cup \{n+1\}$, as desired.

  By construction, the image of s under F' is obtained by adding a new move, $n+1$, to $F'(s')$ (i.e., to V' by induction hypothesis), with $(n+1) \cdot t = n+1$, of the form $@_{S_{s',n+1},k',m}$ for some valid k'. The equation agrees with V, so by canonicity it is enough to show $k = k'$.

  Again, there are two cases:
  - if $f(n+1) \in \sqrt{A}$, then $k' = 1$ but also $k = 1$;

– otherwise $k' = 1 + L_s(n+1)$ by definition of F'.

In the latter case, recall that $L_s(n+1)$ is the index of $\varphi(n+1)$ in the sequence of odd moves in $\lceil s \rceil_{n+1}$. As we saw above, $\varphi(n+1) = 2k-3$. But, s being a TO-preview, we have for all l that $I^s_l = l$, so $L_s(n+1) = k-1$ and $k' = k$, as desired. This achieves the proof that $F'(s) = V$.

□

6. Conclusion and perspectives

We have introduced a notion of signature for the sheaf-based approach to concurrent game semantics [19, 7]. We have then provided sufficient conditions for the pseudo double category $\mathbb{D}(\mathsf{S})$ associated to a signature S to be fibred. We have also instantiated this framework by defining a signature S_{HON} for standard HON games. We have related the obtained pseudo double category to Tsukada and Ong's categories of views and plays, by defining a full embedding of TO-plays into ours (Theorem 144) which restricts to an equivalence on views (Theorem 151). The theory of exact squares has then entailed the following correctness result: in both settings, right Kan extension provides a functor from innocent strategies as presheaves on views to innocent strategies as presheaves on plays; Corollary 116 shows that both processes coincide up to the above full embeddings, and records our main result that restriction along F forms a functor from our innocent strategies to Tsukada and Ong's, which restricts to an equivalence on innocent strategies. In passing, we have established characterisations of our categories of views and plays for HON games as subcategories of slices of the presheaf category $\widehat{\mathbb{L}}$.

A lot of future work remains. First of all, as observed in Remark 11, the notion of view used in Section 3.1 is not entirely in line with our previous models. Mimicking those models would lead to adding a new seed cospan

$$(B_1, \ldots, B_m, A \cdot q \vdash) \xrightarrow{s} \lambda_{S,q} \xleftarrow{t} S,$$

which presents the inconvenience of breaking persistence, one of our sufficient conditions for $\mathbb{D}(\mathsf{S}_{HON})$ to be fibred. In fact, in this case, one can exhibit a counterexample: consider restricting the above cospan along $S + A \to S$ (where, recall, A is the conclusion of S). Indeed, with our current version of S_{HON}, such a move does not have any cartesian lifting along its sub-position A. A first open problem is thus to extend the framework to work around this difficulty.

A second important open issue lies in extending the correspondence exhibited in Section 5 to account for composition, starting with so-called *interaction sequences.*

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